

# The cut locus and distance function from a closed subset of a Finsler manifold <sup>\*†</sup>

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## Abstract

We characterize the differentiable points of the distance function from a closed subset  $N$  of an arbitrary dimensional Finsler manifold in terms of the number of  $N$ -segments. In the case of a 2-dimensional Finsler manifold, we prove the structure theorem of the cut locus of a closed subset  $N$ , namely that it is a local tree, it is made of countably many rectifiable Jordan arcs except for the endpoints of the cut locus and that an intrinsic metric can be introduced in the cut locus and its intrinsic and induced topologies coincide.

## 1 Introduction

Since H. Poincaré introduced the notion of the cut locus in 1905, the cut locus of a point or a submanifold in a Riemannian manifold has been investigated by many researchers ([B]). In spite of this fact, any structure theorem of the cut locus has not been established yet except for special Riemannian manifolds. The main difficulty of formulating and proving such a theorem lies in the fact that the cut locus can be as complicated as a fractal set. In fact, Gluck and Singer ([GS]) constructed a smooth 2-sphere of revolution with positive Gaussian curvature admitting a point whose cut point is non-triangulable. Moreover, it is conjectured that the Hausdorff dimension of the cut locus would be a non-integer, if the manifold is not smooth enough.

However, it has been shown that the Hausdorff dimension of the cut locus of a point is an integer for a smooth Riemannian manifold (see [ITd]) and that the distance function to the cut locus of a closed submanifold is locally Lipschitz for a smooth Riemannian manifold as well as for the Finslerian case (see [IT], [LN]). Hence, the cut locus has weak differentiability.

Nevertheless, if we restrict ourselves to a surface, the structure theorem for the Riemannian cut locus has been established. Indeed, the detailed structure of the cut locus of a point or a smooth Jordan arc in a Riemannian 2-manifold have been thoroughly investigated (see [SST], [H]). For example, Hebda proved in [H] that the cut locus  $C_p$  of a point  $p$  in a complete 2-dimensional Riemannian manifold has a local-tree structure and that any two cut points of  $p$  can be joined by a rectifiable arc in the cut locus  $C_p$ .

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if these two cut points are in the same connected component. Here, a topological space  $X$  is called a *local tree* if for any  $x \in X$  and any neighborhood  $U$  of  $x$ , there exists an open neighborhood  $V \subset U$  of  $x$  such that any two points in  $V$  can be joined by a unique continuous arc.

In the present paper, the structure theorem of the cut locus of a closed subset of a Finsler surface will be proved. It should be noted that the investigations of the cut locus of a closed subset are scarce even in the case of a Riemannian manifold. We will also investigate the differentiability of the distance function of a closed subset of an arbitrary dimensional Finsler manifold.

It is well-known that the differentiability of the distance function is closely related to the cut locus. For example, it is known that the squared distance function from a point  $p$  in a complete Riemannian manifold is differentiable at a point  $q$  if and only if there exists a unique minimal geodesic segment joining  $p$  to  $q$  and that the squared distance function is smooth outside of the cut locus of  $p$ . One of our main theorems (Theorem A) generalizes the facts above.

Let  $N$  be a closed subset of a forward complete Finsler manifold  $(M, F)$ . Roughly speaking, the Finsler manifold is a differentiable manifold with a norm on each tangent space. The precise definition of the Finsler manifold and some necessary fundamental notation and formulas will be reviewed later.

A (unit speed) geodesic segment  $\gamma : [0, a] \rightarrow M$  is called an *N-segment* if  $d(N, \gamma(t)) = t$  holds on  $[0, a]$ , where  $d(N, p) := \min\{d(q, p) \mid q \in N\}$  for each point  $p \in M$ . If a non-constant unit speed  $N$ -segment  $\gamma : [0, a] \rightarrow M$  is maximal as an  $N$ -segment, then the endpoint  $\gamma(a)$  is called a *cut point* of  $N$  along  $\gamma$ . The *cut locus*  $C_N$  of  $N$  is the set of all cut points along all non-constant  $N$ -segments.

One of our main theorems is on the distance function from a closed subset of a Finsler manifold. The research of the distance function  $d_N(\cdot) := d(N, \cdot)$  from the closed subset  $N$  is fundamental in the study of variational problems. For example, the viscosity solution of the Hamilton-Jacobi equation is given by the flow of the gradient vector of the distance function  $d_N$ , when  $N$  is the smooth boundary of a relatively compact domain in Euclidean space (see [LN]).

Although we do not assume any differentiability condition for the closed subset  $N \subset M$ , we prove the following remarkable result.

**Theorem A** *Let  $N$  be a closed subset of a forward complete arbitrary dimensional Finsler manifold  $(M, F)$ . Then, the distance function  $d_N$  from the subset  $N$  is differentiable at a point  $q \in M \setminus N$  if and only if  $q$  admits a unique  $N$ -segment.*

Theorem B and Theorem C are our main theorems on the cut locus. The theorems corresponding to Theorems B and C have been proved in [ShT] for the cut locus of a compact subset of an Alexandrov surface. We should point out that the Toponogov comparison theorem was a key tool for proving main theorems in [ShT], but the Toponogov comparison theorem does not hold for Finsler manifolds. Hence, completely different proofs will be given to Theorems B and C.

**Theorem B** *Let  $N$  be a closed subset of a forward complete 2-dimensional Finsler manifold  $(M, F)$ . Then, the cut locus  $C_N$  of  $N$  satisfies the following properties:*

1.  $C_N$  is a local tree and any two cut points on the same connected component of  $C_N$  can be joined by a rectifiable curve in  $C_N$ .
2. The topology of  $C_N$  induced from the intrinsic metric  $\delta$  coincides with the topology induced from  $(M, F)$ .
3. The space  $C_N$  with the intrinsic metric  $\delta$  is forward complete.
4. The cut locus  $C_N$  is a union of countably many Jordan arcs except for the endpoints of  $C_N$ .

**Theorem C** *There exists a set  $\mathcal{E} \subset [0, \sup d_N)$  of measure zero with the following properties:*

1. For each  $t \in (0, \sup d_N) \setminus \mathcal{E}$ , the set  $d_N^{-1}(t)$  consists of locally finitely many mutually disjoint arcs. In particular, if  $N$  is compact, then  $d_N^{-1}(t)$  consists of finitely many mutually disjoint circles.
2. For each  $t \in (0, \sup d_N) \setminus \mathcal{E}$ , any point  $q \in d_N^{-1}(t)$  admits at most two  $N$ -segments.

**Remark 1.1** Notice that the cut locus of a closed subset is not always closed (see Example 2.7), but the space  $C_N$  with the intrinsic metric  $\delta$  is forward complete for any closed subset of a forward complete Finsler surface. In the case where  $N$  is a compact subset of an Alexandrov surface, all claims in Theorems B and C were proved except for the third claim of Theorem B.

Let us recall that a *Finsler manifold*  $(M, F)$  is an  $n$ -dimensional differential manifold  $M$  endowed with a norm  $F : TM \rightarrow [0, \infty)$  such that

1.  $F$  is positive and differentiable on  $\widetilde{TM} := TM \setminus \{0\}$ ;
2.  $F$  is 1-positive homogeneous, i.e.,  $F(x, \lambda y) = \lambda F(x, y)$ ,  $\lambda > 0$ ,  $(x, y) \in TM$ ;
3. the Hessian matrix  $g_{ij}(x, y) := \frac{1}{2} \frac{\partial^2 F^2}{\partial y^i \partial y^j}$  is positive definite on  $\widetilde{TM}$ .

Here  $TM$  denotes the tangent bundle over the manifold  $M$ . The Finsler structure is called *absolute homogeneous* if  $F(x, -y) = F(x, y)$  because this leads to the homogeneity condition  $F(x, \lambda y) = |\lambda| F(x, y)$ , for any  $\lambda \in \mathbb{R}$ .

By means of the Finsler fundamental function  $F$  one defines the *indicatrix bundle* (or the *Finslerian unit sphere bundle*) by  $SM := \cup_{x \in M} S_x M$ , where  $S_x M := \{y \in M \mid F(x, y) = 1\}$ .

On a Finsler manifold  $(M, F)$  one can define the integral length of curves as follows. Let  $\gamma : [a, b] \rightarrow M$  be a regular piecewise  $C^\infty$ -curve in  $M$ , and let  $a := t_0 < t_1 < \dots < t_k := b$  be a partition of  $[a, b]$  such that  $\gamma|_{[t_{i-1}, t_i]}$  is smooth for each interval  $[t_{i-1}, t_i]$ ,  $i \in \{1, 2, \dots, k\}$ . The *integral length* of  $\gamma$  is given by

$$L(\gamma) := \sum_{i=1}^k \int_{t_{i-1}}^{t_i} F(\gamma(t), \dot{\gamma}(t)) dt, \quad (1.1)$$

where  $\dot{\gamma} = \frac{d\gamma}{dt}$  is the tangent vector along the curve  $\gamma|_{[t_{i-1}, t_i]}$ . For such a partition, let us consider a regular piecewise  $C^\infty$ -map

$$\bar{\gamma} : (-\varepsilon, \varepsilon) \times [a, b] \rightarrow M, \quad (u, t) \mapsto \bar{\gamma}(u, t) \quad (1.2)$$

such that  $\bar{\gamma}|_{(-\varepsilon, \varepsilon) \times [t_{i-1}, t_i]}$  is smooth for all  $i \in \{1, 2, \dots, k\}$ , and  $\bar{\gamma}(0, t) = \gamma(t)$ . Such a curve is called a regular piecewise  $C^\infty$ -variation of the base curve  $\gamma(t)$ , and the vector field  $U(t) := \frac{\partial \bar{\gamma}}{\partial u}(0, t)$  is called the *variational vector field* of  $\bar{\gamma}$ . The integral length  $\mathcal{L}(u)$  of  $\bar{\gamma}(u, t)$  will be a function of  $u$ , defined as in (1.1).

By a straightforward computation one obtains

$$\begin{aligned} \mathcal{L}'(0) = & g_{\dot{\gamma}(b)}(\gamma, U)|_a^b + \sum_{i=1}^k \left[ g_{\dot{\gamma}(t_i^-)}(\dot{\gamma}(t_i^-), U(t_i)) - g_{\dot{\gamma}(t_i^+)}(\dot{\gamma}(t_i^+), U(t_i)) \right] \\ & - \int_a^b g_{\dot{\gamma}}(D_{\dot{\gamma}}\dot{\gamma}, U)dt, \end{aligned} \quad (1.3)$$

where  $D_{\dot{\gamma}}$  is the covariant derivative along  $\gamma$  with respect to the Chern connection and  $\gamma$  is arc length parametrized (see [BCS], p. 123, or [S], p. 77 for details of this computation as well as for the basis on Finslerian connections).

A regular piecewise  $C^\infty$ -curve  $\gamma$  on a Finsler manifold is called a *geodesic* if  $\mathcal{L}'(0) = 0$  for all piecewise  $C^\infty$ -variations of  $\gamma$  that keep its ends fixed. In terms of Chern connection a constant speed geodesic is characterized by the condition  $D_{\dot{\gamma}}\dot{\gamma} = 0$ .

Let now  $\gamma : [a, b] \rightarrow M$  be a unit speed geodesic and  $\sigma : (-\varepsilon, \varepsilon) \rightarrow M$  a  $C^\infty$ -curve such that  $\sigma(0) = \gamma(b)$ . If one considers a  $C^\infty$ -variation  $\bar{\gamma} : (-\varepsilon, \varepsilon) \times [a, b] \rightarrow M$  with one end fixed and another one on the curve  $\sigma$ , i.e.

$$\bar{\gamma}(u, a) = \gamma(a), \quad \bar{\gamma}(u, b) = \sigma(u), \quad (1.4)$$

then formula (1.3) implies that the integral length  $\mathcal{L}(u)$  of the curve  $\bar{\gamma}_u(t) := \bar{\gamma}(u, t)$ ,  $t \in [a, b]$  satisfies the *first variation formula* ([S], p. 78):

$$\mathcal{L}'(0) = g_{\dot{\gamma}(b)}(\dot{\gamma}(b), \dot{\sigma}(0)). \quad (1.5)$$

This formula is fundamental for our present study.

Using the integral length of a curve, one can define the Finslerian distance between two points on  $M$ . For any two points  $p, q$  on  $M$ , let us denote by  $\Omega_{p,q}$  the set of all piecewise  $C^\infty$ -curves  $\gamma : [a, b] \rightarrow M$  such that  $\gamma(a) = p$  and  $\gamma(b) = q$ . The map

$$d : M \times M \rightarrow [0, \infty), \quad d(p, q) := \inf_{\gamma \in \Omega_{p,q}} L(\gamma) \quad (1.6)$$

gives the *Finslerian distance* on  $M$ . It can be easily seen that  $d$  is in general a quasi-distance, i.e., it has the properties

1.  $d(p, q) \geq 0$ , with equality if and only if  $p = q$ ;
2.  $d(p, q) \leq d(p, r) + d(r, q)$ , with equality if and only if  $r$  lies on a minimal geodesic segment joining from  $p$  to  $q$  (triangle inequality).

In the case where  $(M, F)$  is absolutely homogeneous, the symmetry condition  $d(p, q) = d(q, p)$  holds and therefore  $(M, d)$  is a genuine metric space. We do not assume this symmetry condition in the present paper.

Let us also recall that for a forward complete Finsler space  $(M, F)$ , the exponential map  $\exp_p : T_p M \rightarrow M$  at an arbitrary point  $p \in M$  is a surjective map (see [BCS], p. 152 for details). This will be always assumed in the present paper.

A unit speed geodesic on  $M$  with initial conditions  $\gamma(0) = p \in M$  and  $\dot{\gamma}(0) = T \in S_p M$  can be written as  $\gamma(t) = \exp_p(tT)$ . Even though the exponential map is quite similar with the correspondent notion in Riemannian geometry, we point out two distinguished properties (see [BCS], p. 127 for details):

1.  $\exp_x$  is only  $C^1$  at the zero section of  $TM$ , i.e. for each fixed  $x$ , the map  $\exp_x y$  is  $C^1$  with respect to  $y \in T_x M$ , and  $C^\infty$  away from it. Its derivative at the zero section is the identity map (Whitehead);
2.  $\exp_x$  is  $C^2$  at the zero section of  $TM$  if and only if the Finsler structure is of Berwald type. In this case  $\exp$  is actually  $C^\infty$  on entire  $TM$  (Akbar-Zadeh).

## 2 The distance function from a closed subset

Let  $N$  a closed subset of a forward complete Finsler manifold  $(M, F)$ . For each point  $p \in M \setminus N$ , we denote by  $\Gamma_N(p)$  the set of all unit speed  $N$ -segments to  $p$ . Here a unit speed geodesic segment  $\gamma : [0, a] \rightarrow M$  is called an  $N$ -segment if  $d(N, \gamma(t)) = t$  holds on  $[0, a]$ , where  $d(N, p) := \min\{d(q, p) \mid q \in N\}$  for each point  $p \in M$ . If a non-constant unit speed  $N$ -segment  $\gamma : [0, a] \rightarrow M$  is maximal as an  $N$ -segment, then the endpoint  $\gamma(a)$  is called a *cut point* of  $N$  along  $\gamma$ . The *cut locus*  $C_N$  is the set of all cut points along all non-constant  $N$ -segments. Hence  $C_N \cap N = \emptyset$ . Notice that there might exists a sequence of cut points convergent to  $N$  if  $N$  is not a submanifold.

**Remark 2.1** We discuss here only the forward complete case. Let us point out that in the Finsler case, unlikely the Riemannian counterpart, forward completeness is not equivalent to backward one, except the case when  $M$  is compact.

First, two versions of the first variation formula for the distance function from the closed set  $N$  will be stated and proved. These formulas are fundamental for the study of the cut locus hereafter. In Proposition 2.6, as an application of the first variation formula, it is proved that the subset of the cut locus of  $N$  that consists of all cut points of  $N$  admitting at least two  $N$ -segments is dense in  $C_N$ .

The following proposition was proved in [IT] for the distance function from a closed submanifold of a complete Riemannian manifold. Since the distance function on a Finsler manifold is not always symmetric, we have two versions of the first variation formula in our case.

**Proposition 2.2 (Generalized first variation formula, forward version)**

Let  $\{\gamma_i : [0, l_i] \rightarrow M\}$  be a convergent sequence of  $N$ -segments in an  $n$ -dimensional Finsler manifold  $M$ . If the limit

$$v^f := \lim_{i \rightarrow \infty} \frac{1}{F(\exp_x^{-1}(\gamma_i(l_i)))} \exp_x^{-1}(\gamma_i(l_i)), \quad (2.1)$$

exists, then

$$g_{w_\infty}(v^f, w_\infty) = \min\{g_w(v^f, w) \mid w \text{ is the unit velocity tangent vector at } x \text{ of } \gamma \in \Gamma_N(x)\}. \quad (2.2)$$

Moreover,

$$\lim_{i \rightarrow \infty} \frac{d(N, \gamma_i(l_i)) - d(N, x)}{d(x, \gamma_i(l_i))} = g_{w_\infty}(v^f, w_\infty) \quad (2.3)$$

holds.

Here  $x := \lim_{i \rightarrow \infty} \gamma_i(l_i)$ ,  $w_\infty := \lim_{i \rightarrow \infty} \dot{\gamma}_i(l_i) \in T_x M$ , and  $\exp_x^{-1}$  denotes the local inverse map of the exponential map  $\exp_x$  around the zero vector.

*Proof.* Let  $\sigma_i : [0, d(x, \gamma_i(l_i))] \rightarrow M$  denote the unit speed minimal geodesic segment emanating from  $x$  to  $\gamma_i(l_i)$ , and hence

$$\dot{\sigma}_i(0) = \frac{1}{F(\exp_x^{-1}(\gamma_i(l_i)))} \exp_x^{-1}(\gamma_i(l_i)). \quad (2.4)$$

Let us choose a positive constant  $\delta$  in such a way that  $\gamma(l - \delta)$  is a point of a strongly convex ball around  $x$ . Here  $\gamma := \lim_{i \rightarrow \infty} \gamma_i$  and  $l := \lim_{i \rightarrow \infty} l_i$ .

By the triangle inequality

$$d(N, x) \leq d(N, \gamma_i(l - \delta)) + d(\gamma_i(l - \delta), x),$$

and hence, we obtain

$$d(N, \gamma_i(l_i)) - d(N, x) \geq d(\gamma_i(l - \delta), \gamma_i(l_i)) - d(\gamma_i(l - \delta), x). \quad (2.5)$$

If we apply the Taylor expansion formula for the function  $f(t) := d(\gamma_i(l - \delta), \sigma_i(t))$ , it follows from the first variation formula (1.5) that there exists a positive constant  $C$  such that for any  $i$  and any sufficiently small  $|t|$

$$d(\gamma_i(l - \delta), \sigma_i(t)) \geq d(\gamma_i(l - \delta), x) + g_{w_i}(w_i, \dot{\sigma}_i(0))t - Ct^2, \quad (2.6)$$

where  $w_i$  denotes the unit velocity vector at  $x$  of the minimal geodesic segment joining from  $\gamma_i(l - \delta)$  to  $x$ . Thus, we obtain, by (2.5) and (2.6)

$$\liminf_{i \rightarrow \infty} \frac{d(N, \gamma_i(l_i)) - d(N, x)}{d(x, \gamma_i(l_i))} \geq \liminf_{i \rightarrow \infty} g_{w_i}(w_i, \dot{\sigma}_i(0)). \quad (2.7)$$

Since  $\lim_{i \rightarrow \infty} \gamma_i = \gamma$ , we have

$$\lim_{i \rightarrow \infty} w_i = w_\infty. \quad (2.8)$$

From (2.1), (2.4), (2.7), and (2.8), it follows

$$\liminf_{i \rightarrow \infty} \frac{d(N, \gamma_i(l_i)) - d(N, x)}{d(x, \gamma_i(l_i))} \geq g_{w_\infty}(w_\infty, v^f). \quad (2.9)$$

Let  $\beta : [0, l] \rightarrow M$  be any unit speed  $N$ -segment in  $\Gamma_N(x)$ . The triangle inequality again gives

$$d(N, \gamma_i(l_i)) \leq d(N, \beta(l - \delta)) + d(\beta(l - \delta), \gamma_i(l_i)), \quad (2.10)$$

where the positive constant  $\delta$  is chosen in such a way that  $\beta(l - \delta)$  lies in a strongly convex ball at  $x$ .

The relation (2.10) implies

$$d(N, \gamma_i(l_i)) - d(N, x) \leq d(\beta(l - \delta), \gamma_i(l_i)) - d(\beta(l - \delta), x) \quad (2.11)$$

and from the Taylor expansion and the first variation formula (1.5) it results that there exists a positive constant  $C$  such that

$$d(\beta(l - \delta), \gamma_i(l_i)) - d(\beta(l - \delta), x) \leq g_{w(\beta)}(w(\beta), \dot{\sigma}_i(0))d(x, \gamma_i(l_i)) + Cd(x, \gamma_i(l_i))^2 \quad (2.12)$$

for any  $i$ , where  $w(\beta) := \dot{\beta}(l)$ . Hence, for any  $N$ -segment  $\beta \in \Gamma_N(x)$ , we have

$$\limsup_{i \rightarrow \infty} \frac{d(N, \gamma_i(l_i)) - d(N, x)}{d(x, \gamma_i(l_i))} \leq \lim_{i \rightarrow \infty} g_{w(\beta)}(w(\beta), \dot{\sigma}_i(0)) = g_{w(\beta)}(w(\beta), v^f). \quad (2.13)$$

In particular, we obtain

$$\limsup_{i \rightarrow \infty} \frac{d(N, \gamma_i(l_i)) - d(N, x)}{d(x, \gamma_i(l_i))} \leq g_{w_\infty}(w_\infty, v^f). \quad (2.14)$$

Now, the relation (2.3) follows from (2.9) and (2.14), while (2.2) is implied by (2.9) and (2.13).  $\square$

**Proposition 2.3 (Generalized first variation formula, backward version)**

Let  $\{\gamma_i : [0, l_i] \rightarrow M\}$  be a convergent sequence of  $N$ -segments in an  $n$ -dimensional Finsler manifold  $M$ . If the limit

$$v^b := \lim_{i \rightarrow \infty} \frac{1}{F(\exp_{\gamma_i(l_i)}^{-1}(x))} \exp_{\gamma_i(l_i)}^{-1}(x) \quad (2.15)$$

exists, then

$$g_{w_\infty}(-v^b, w_\infty) = \min\{g_w(-v^b, w) \mid w \text{ is the unit velocity tangent vector at } x \text{ of } \gamma \in \Gamma_N(x)\}.$$

Moreover,

$$\lim_{i \rightarrow \infty} \frac{d(N, \gamma_i(l_i)) - d(N, x)}{d(\gamma_i(l_i), x)} = g_{w_\infty}(-v^b, w_\infty) \quad (2.16)$$

holds.

Here  $x := \lim_{i \rightarrow \infty} \gamma_i(l_i)$  and  $w_\infty := \lim_{i \rightarrow \infty} \dot{\gamma}_i(l_i)$ .



*Proof.* The proof is similar to that of Proposition 2.2, if we apply the Taylor expansion for the functions  $d(\gamma_i(l - \delta), \sigma_i(t))$  and  $d(\beta(l - \delta), \sigma_i(t))$ . Here  $\sigma_i : [-d(\gamma_i(l_i), x), 0] \rightarrow M$  denotes the minimal geodesic segment emanating from  $\gamma_i(l_i)$  to  $x$ .  $\square$

The following theorem gives a necessary and sufficient condition for the distance function from a closed subset to be differentiable at a point. This theorem corresponds to the theorem of the differentiability of a Busemann function (see [KTI]).

**Theorem 2.4** *Let  $N$  be a closed subset of a forward complete  $n$ -dimensional Finsler manifold  $M$ . Then, the distance function  $d_N(\cdot) := d(N, \cdot)$  is differentiable at a point  $q \in M \setminus N$  if and only if there exists a unique  $N$ -segment to  $q$ . Furthermore, the differential  $(dd_N)_q$  of  $d_N$  at a differentiable point  $q \in M \setminus N$  satisfies that*

$$(dd_N)_q(v) = g_X(X, v)$$

for any  $v \in T_q M$ . Here  $X$  denotes the velocity vector at  $q$  of the unique  $N$ -segment to  $q$ .

*Proof.* Suppose that a point  $q \in M \setminus N$  admits a unique  $N$ -segment  $\alpha : [0, l] \rightarrow M$ . Let  $v$  be any tangent vector with  $F(v) = 1$ . We obtain, by Proposition 2.2,

$$\lim_{t \searrow 0} \frac{(d_N \circ \exp_q)(tv) - (d_N \circ \exp_q)(O_q)}{t} = g_{\dot{\alpha}(l)}(\dot{\alpha}(l), v).$$

Hence, by Lemma 2.5,  $d_N \circ \exp_q$  is differentiable at the zero vector  $O_q$ . This implies that  $d_N$  is differentiable at  $q$ , since  $(d \exp_q)_{O_q}$  is the identity map on the tangent space  $T_q M$  at  $q$ . Suppose next that  $d_N$  is differentiable at a point  $q \in M \setminus N$ , and let  $\alpha : [0, l] \rightarrow M$  be a unit speed  $N$ -segment to  $q$ . It is clear that

$$\lim_{t \nearrow l} \frac{d_N(\alpha(t)) - d_N(q)}{t - l} = 1.$$

Since  $d_N(\alpha(t))$  is differentiable at  $t = 0$ , we obtain

$$\lim_{t \searrow l} \frac{d_N(\alpha(t)) - d_N(q)}{t - l} = 1. \quad (2.17)$$

Choose a decreasing sequence  $\{t_i\}$  convergent to  $l$  in such a way that the sequence of  $N$ -segments to  $\alpha(t_i)$  has a unique limit  $N$ -segment  $\beta$ . Here the  $N$ -segment  $\alpha$  is assumed to be extended as the geodesic on  $[0, \infty)$ . From Proposition 2.2 and (2.17) it follows that

$$1 = \lim_{i \rightarrow \infty} \frac{d_N(\alpha(t_i)) - d_N(q)}{t_i - l} = g_{\dot{\beta}(l)}(\dot{\beta}(l), \dot{\alpha}(l))$$

and

$$g_{\dot{\beta}(l)}(\dot{\beta}(l), \dot{\alpha}(l)) = \min\{g_{\dot{\gamma}(l)}(\dot{\gamma}(l), \dot{\alpha}(l)) \mid \gamma \in \Gamma_N(q)\}.$$

Hence,  $g_{\dot{\gamma}(l)}(\dot{\gamma}(l), \dot{\alpha}(l)) \geq 1$ , for any  $\gamma \in \Gamma_N(q)$ . Therefore, by Lemma 1.2.3 in [S],  $\dot{\gamma}(l) = \dot{\alpha}(l)$  for any  $\gamma \in \Gamma_N(q)$ , and  $q$  admits a unique  $N$ -segment.  $\square$



**Lemma 2.5** *Let  $f : U \rightarrow R$  be a Lipschitz function on a open convex subset around the zero vector  $O$  of a Minkowski space  $(V, F)$  with a Minkowski norm  $F$ . Suppose that there exists a linear function  $\omega : V \rightarrow R$  such that for each  $e \in F^{-1}(1)$ ,*

$$\lim_{\lambda \searrow 0} \frac{f(\lambda e) - f(O) - \omega(\lambda e)}{\lambda} = 0. \quad (2.18)$$

*Then,*

$$\lim_{F(v) \rightarrow 0} \frac{f(v) - f(O) - \omega(v)}{F(v)} = 0, \quad (2.19)$$

*i.e.,  $f$  is differentiable at the zero vector  $O$ , and its differential at  $O$  is  $\omega$ .*

*Proof.* Choose any positive number  $\epsilon$  and fix it. Since  $F^{-1}(1)$  is compact, we may choose finitely many elements  $e_1, \dots, e_k$  of  $F^{-1}(1)$  in such a way that for any  $e \in F^{-1}(1)$ , there exists some  $e_i$  satisfying

$$F(e - e_i) < \frac{\epsilon}{3L} \quad \text{and} \quad |\omega(e - e_i)| < \epsilon/3. \quad (2.20)$$

Here  $L$  denotes a Lipschitz constant of the function  $f$ . Let  $v$  be any non-zero vector of  $U$  and choose any  $e_i$ . By the triangle inequality,

$$|f(v) - f(O) - \omega(v)| \leq |f(v) - f(\lambda e_i)| + |f(\lambda e_i) - f(O) - \omega(\lambda e_i)| + |\omega(v) - \omega(\lambda e_i)|, \quad (2.21)$$

where  $\lambda := F(v)$ . Since  $f$  is a Lipschitz function with a Lipschitz constant  $L$ ,

$$|f(v) - f(\lambda e_i)| \leq LF(v - \lambda e_i) = \lambda LF(e(v) - e_i),$$

where  $e(v) := \frac{v}{\lambda} \in F^{-1}(1)$ . Hence, by means of (2.18) and (2.21), we get

$$\limsup_{F(v) \rightarrow 0} \frac{|f(v) - f(O) - \omega(v)|}{F(v)} \leq L \limsup_{\lambda \searrow 0} |F(e(v) - e_i)| + \limsup_{\lambda \searrow 0} |\omega(e(v) - e_i)|.$$

Here, by choosing  $e_i$  so as to satisfy (2.20), we get

$$\limsup_{F(v) \rightarrow 0} \frac{|f(v) - f(O) - \omega(v)|}{F(v)} \leq \epsilon.$$

Since  $\epsilon$  is arbitrarily chosen, relation (2.19) follows. □

Let us recall that the cut locus  $C_N$  is the set of all endpoints of non-constant maximal  $N$ -segments and that each element of  $C_N$  is called a cut point of  $N$ .

The following proposition was proved by Bishop ([Bh]) for the cut locus of a point in a Riemannian manifold. However, the proof of the following proposition is direct, and hence completely different from the one by Bishop.

**Proposition 2.6** *Let  $N$  be a closed subset of a forward complete  $n$ -dimensional Finsler manifold  $M$ . Then the subset of  $C_N$ , which consists of all cut points of  $N$  admitting (at least) two  $N$ -segments, is dense in the cut locus of  $N$ .*

*Proof.* Let  $p$  be a cut point of  $N$  which admits a unique  $N$ -segment. Suppose that there exists an open ball  $B_{\delta_1}(p)$  each element of which admits a unique  $N$ -segment. From Theorem 2.4 it follows that the distance function  $d_N|_{B_{\delta_1}(p)}$  is a  $C^1$ -function and has no critical points. Hence, by the inverse function theorem, there exists a  $C^1$ -diffeomorphism  $\varphi : (l - 2\delta_2, l + 2\delta_2) \times U_{\delta_3} \rightarrow \varphi((l - 2\delta_2, l + 2\delta_2) \times U_{\delta_3}) \subset B_{\delta_1}(p)$  such that  $\varphi(l, O) = p$ , and  $d(N, \varphi(t, q)) = t$  on  $(l - 2\delta_2, l + 2\delta_2) \times U_{\delta_3}$ . Here  $l := d(N, p)$ , and  $U_{\delta_3}$  denotes the open ball of radius  $\delta_3$  centered at the origin  $O$  in  $R^{n-1}$ . For each point  $x \in U_{\delta_3}$ , let  $\gamma_x : [0, l + \delta_2] \rightarrow M$  denote the  $N$ -segment passing through  $\varphi(l, x)$ . Since the set  $\bigcup_{x \in U_\epsilon} \gamma_x[0, l + \delta_2]$  is a neighborhood of  $p$  for each  $\epsilon \in (0, \delta_3)$ , the family  $\{\gamma_x|_{[0, l + \delta_2]}\}$  of  $N$ -segments shrinks to an  $N$ -segment  $\gamma : [0, l + \delta_2] \rightarrow M$  passing through  $p$  when  $\epsilon$  goes to zero. This is a contradiction.  $\square$

It is known that the cut locus of a point in a complete Finsler manifold is closed (see for example [BCS]). The following example shows that in general the cut locus of a closed subset in Euclidean plane is not closed.

**Example 2.7** Choose any strictly decreasing sequence  $\{\theta_n\}$  with  $\theta_1 \in (0, \pi)$  which is convergent to zero. Let  $D$  denote the closed ball with radius 1 centered at the origin of Euclidean plane  $E^2$  endowed with the standard Euclidean norm and let  $B_n$  be the open ball with radius 1 centered at  $q_n$ , for each  $n = 1, 2, 3, \dots$ . Here  $q_n \notin D$  denotes the center of the circle with radius 1 passing through two points  $(\cos \theta_n, \sin \theta_n)$  and  $(\cos \theta_{n+1}, \sin \theta_{n+1})$ . A closed subset  $N$  of Euclidean plane is defined by

$$N := D \setminus \bigcup_{n=1}^{\infty} B_n.$$

It is trivial to see that the sequence  $\{q_n\}$  of cut points of  $N$  converges to the point  $(x, y) = (2, 0)$ . On the other hand, the point  $(x, y) = (2, 0)$  lies on the  $N$ -segment  $\{(x, 0) \mid 1 \leq x \leq 3\}$ . This implies that the cut locus of the set  $N$  is not closed in  $E^2$ .

### 3 The cut locus is a local tree

From now on  $N$  denotes a closed subset of a forward complete 2-dimensional Finsler manifold  $(M, F)$ . For each point  $p \in M \setminus N$ , we denote by  $\Gamma_N(p)$  the set of all unit speed  $N$ -segments to  $p$ , and by  $B_\delta(q)$  the forward ball

$$B_\delta(q) := \{r \in M \mid d(q, r) < \delta\},$$

centered at a point  $q \in M$  and of radius  $\delta$ .

Let  $x$  be a cut point of  $N$ . Choose a small  $\delta_0 > 0$  (to be fixed) in such a way that  $B_{4\delta_0}(x)$  is a strongly convex neighborhood at  $x$ . For any  $y \in C_N \cap B_{\delta_0}(x)$ , each connected component of

$$B_{3\delta_0}(x) \setminus \{\gamma[0, d(N, y)] \mid \gamma \in \Gamma_N(y)\}$$

is called a *sector* at  $y$ .

Choose any distinct two cut points  $y_0$  and  $y_1$  of  $N$  from  $B_{\delta_0}(x)$ . One can easily see that any  $\gamma \in \Gamma_N(y_0)$  does not pass through  $y_1$ . Hence there exists a unique sector  $\Sigma_{y_0}(y_1)$  at  $y_0$  containing  $y_1$ . Let  $\Sigma_{y_1}(y_0)$  denote the sector at  $y_1$  containing  $y_0$ . Since each  $N$ -segment to a point in  $B_{\delta_0}(x)$  intersects  $S_{2\delta_0}(x) := \{q \in M | d(x, q) = 2\delta_0\}$  exactly once, the set

$$W(y_0, y_1) := \Sigma_{y_0}(y_1) \cap \Sigma_{y_1}(y_0) \cap B_{2\delta_0}(x)$$

is a 2-disc domain. Furthermore, there exist exactly two open subarcs  $I$  and  $J$  of  $S_{2\delta_0}(x)$  cut off by  $N$ -segments in  $\Gamma_N(y_0)$  or  $\Gamma_N(y_1)$ . If  $\Gamma_N(y_0)$  or  $\Gamma_N(y_1)$  consists of a single  $N$ -segment, then  $I$  and  $J$  have a common end point. Notice that for each point  $r \in W(y_0, y_1)$ , any  $N$ -segment to  $r$  meets  $I$  or  $J$ . Let  $W_I(y_0, y_1)$  (respectively  $W_J(y_0, y_1)$ ) denote the set of all points  $r$  in  $W(y_0, y_1)$  which admit an  $N$ -segment intersecting  $I$  (respectively  $J$ ).

**Lemma 3.1** *Neither of  $W_I(y_0, y_1)$  nor  $W_J(y_0, y_1)$  is empty. Moreover, if  $y_0$  and  $y_1$  are sufficiently close each other, then  $W_I(y_0, y_1) \cap W_J(y_0, y_1)$  is a subset of  $B_{\delta_0}(x)$ .*

*Proof.* Let  $\gamma_I$  and  $\gamma_J$  denote the  $N$ -segments in  $\Gamma_N(y_0)$  that form part of the boundary of  $W(y_0, y_1)$ . Here we assume that  $\gamma_I$  (respectively  $\gamma_J$ ) intersects  $S_{2\delta_0}(x)$  at an end point of  $I$  (respectively  $J$ ). Notice that  $\gamma_I = \gamma_J$  holds if and only if  $\Gamma_N(y_0)$  consists of a single element. Take  $t_0 \in (0, d(N, y_0))$  so as to satisfy that  $\gamma_I(t_0)$  and  $\gamma_J(t_0)$  are points in  $B_{\delta_0}(x)$ . Choose strongly convex neighborhoods  $B_\epsilon(\gamma_I(t_0)) (\subset B_{\delta_0}(x))$  and  $B_\epsilon(\gamma_J(t_0)) (\subset B_{\delta_0}(x))$  in such a way that  $B_\epsilon(\gamma_I(t_0)) \cap B_\epsilon(\gamma_J(t_0)) = \emptyset$  if  $\gamma_I \neq \gamma_J$ . It is clear that

$$D_I := W(y_0, y_1) \cap B_\epsilon(\gamma_I(t_0)) \quad \text{and} \quad D_J := W(y_0, y_1) \cap B_\epsilon(\gamma_J(t_0))$$

are disjoint if  $\gamma_I \neq \gamma_J$ .

In the case when  $\gamma_I = \gamma_J$ ,  $D_I$  and  $D_J$  denote the two connected components of  $B_\epsilon(\gamma_I(t_0)) \setminus \gamma_I[0, d(N, y_0)]$ . In this case, we may assume that for each  $t \in I$  sufficiently close to the intersection of  $\gamma_I$  and  $S_{2\delta_0}(x)$ , the minimal geodesic segment from  $t$  to  $\gamma_I(t_0)$  intersects  $D_I$ , but does not intersect  $D_J$ .

Suppose that  $\gamma_I \neq \gamma_J$  and  $W_I(y_0, y_1)$  or  $W_J(y_0, y_1)$  is empty. Without loss of generality, we may assume that  $W_I(y_0, y_1) = \emptyset$ . Choose a sequence  $\{q_n\}$  of points in  $D_I$  converging to  $\gamma_I(t_0)$ . Let  $\alpha$  be a limit  $N$ -segment of the sequence  $\{\alpha_n\}$ , where  $\alpha_n \in \Gamma_N(q_n)$ . Since we have assumed that  $W_I(y_0, y_1)$  is empty, for each  $n$ ,  $\alpha_n$  intersects  $J$ . The  $N$ -segment  $\alpha$  intersects the closure  $\bar{J}$  of  $J$ . Hence  $\gamma_I(t_0)$  admits two  $N$ -segments  $\alpha$  and  $\gamma_I|_{[0, t_0]}$  if  $\gamma_I \neq \gamma_J$ , that is, a contradiction. Therefore, we must have  $\gamma_I = \gamma_J$ .

Choose any point  $q_J$  from  $D_J$ , and fix it. Let  $\alpha_J : [0, d(N, q_J)] \rightarrow M$  be an element of  $\Gamma_N(q_J)$ , and  $\beta$  the unique minimal geodesic segment joining from  $q_J$  to  $\gamma_I(t_0) = \gamma_J(t_0)$ . Since we assumed that  $W_I(y_0, y_1)$  is empty,  $\alpha_J$  intersects  $S_{2\delta_0}(x)$  at a point of  $J$ . Then the three geodesic segments  $\alpha_J$ ,  $\beta$  and  $\gamma_J|_{[0, t_0]}$  bound a 2-disc domain  $D(\alpha_J, \beta)$  together with the subarc  $c$  of  $\bar{J}$  cut off by  $\alpha_J$  and  $\gamma_I = \gamma_J$ . Since we assumed that  $W_I(y_0, y_1)$  is empty, the  $N$ -segment  $\alpha_n$  intersects  $J$  for each  $n$  and the sequence  $\{\alpha_n\}$  converges to  $\gamma_I|_{[0, t_0]}$ . Therefore, for any sufficiently large  $n$ ,  $\alpha_n$  intersects  $c$ , the subarc of  $\bar{J}$ . Hence  $\alpha_n$  passes through the disc domain  $D(\alpha_J, \beta)$ , and intersects  $\beta$  at a point  $p_n \in D_J$ . The subarc  $\gamma_n$  of  $\alpha_n$  with end points  $p_n$  and  $q_n$  is minimal and both end points are in  $B_\epsilon(\gamma_I(t_0))$ . Since  $B_\epsilon(\gamma_I(t_0))$  is a strongly convex ball, the subarc is entirely contained in the ball and

joins  $p_n \in D_J$  to  $q_n \in D_I$ . Hence  $\gamma_n$  meets  $\gamma_I$  at a point in  $B_\epsilon(\gamma_I(t_0))$ . This is again a contradiction, since both  $\alpha_n$  and  $\gamma_I$  are  $N$ -segments. The second claim can be proved by a similar argument as above.  $\square$

**Lemma 3.2** *For each  $x \in C_N$  and each sector  $\Sigma_x$  at  $x$ , there exists a sequence of points in  $\Sigma_x \cap C_N$  convergent to  $x$ .*

*Proof.* Suppose that there exists no cut point of  $N$  in  $B_\epsilon(x) \cap \Sigma_x$  for some sufficiently small positive  $\epsilon$ . Let  $\gamma$  denote an  $N$ -segment to a point in  $\Sigma_x \cap S_{\epsilon/2}(x)$ . Take any  $\delta \in (0, \epsilon/2)$ . For each point  $y \in \Sigma_x \cap S_\delta(x)$ , there exists an  $N$ -segment  $\gamma_y$  to  $y$ . We get a family of  $N$ -segments  $\{\gamma_y\}_{y \in \Sigma_x \cap S_\delta(x)}$ . Since there exists no cut point of  $N$  in  $B_\epsilon(x) \cap \Sigma_x$ , the  $N$ -segment  $\gamma$  is a restriction of  $\gamma_y$  for some  $y \in \Sigma_x \cap S_\delta(x)$ . Since  $\delta$  is chosen arbitrarily small,  $\gamma$  is extensible to an  $N$ -segment to  $x$ , which lies in the sector  $\Sigma_x$ . This contradicts the definition of the sector.  $\square$

**Lemma 3.3**  $W_I(y_0, y_1) \cap W_J(y_0, y_1) \neq \emptyset$ .

*Proof.* It is clear that the set  $W(y_0, y_1)$  is the union of  $W_I(y_0, y_1)$  and  $W_J(y_0, y_1)$ , and that both  $W_I(y_0, y_1)$  and  $W_J(y_0, y_1)$  are relatively closed in  $W(y_0, y_1)$ . Hence  $W_I(y_0, y_1) \cap W_J(y_0, y_1) \neq \emptyset$ , since  $W(y_0, y_1)$  is connected, and neither of  $W_I(y_0, y_1)$  nor  $W_J(y_0, y_1)$  is empty by Lemma 3.1.  $\square$

Let us recall that an injective continuous map from the open interval  $(0, 1)$  or closed interval  $[0, 1]$  of  $\mathbb{R}$  into  $M$  is called a *Jordan arc*. An injective continuous map from a circle  $S^1$  into  $M$  is called a *Jordan curve*. The image of a Jordan arc or a Jordan curve is also called a *Jordan arc* or *Jordan curve*, respectively.

**Lemma 3.4** *Suppose that the cut points  $y_0$  and  $y_1$  of  $N$  are sufficiently close each other, so that  $W_I(y_0, y_1) \cap W_J(y_0, y_1) \subset B_{\delta_0}(x)$ . Then, for each point  $t \in I$ , there exists a unique point  $r \in W_I(y_0, y_1) \cap W_J(y_0, y_1)$  such that there exists a sector at  $r$  containing  $t$  or there exists an  $N$ -segment to  $r$  which passes through the point  $t$ .*

*Proof.* Since  $I$  and the closure  $\bar{I}$  of  $I$  are Jordan arcs, we may assume that  $I = (0, 1)$  and  $\bar{I} = [0, 1]$ . Suppose that there does not exist an  $N$ -segment to a point in  $W_I(y_0, y_1) \cap W_J(y_0, y_1)$  passing through some  $t \in (0, 1) = I$ . Let  $t_+ \in I$  and  $t_- \in I$  denote the minimum and the maximum of the following sets respectively:

$$I_+ := \bigcup_{r \in W_I(y_0, y_1) \cap W_J(y_0, y_1)} \{s \in [t, 1] \mid \text{there exists an element of } \Gamma_N(r) \text{ passing through } s\}$$

$$I_- := \bigcup_{r \in W_I(y_0, y_1) \cap W_J(y_0, y_1)} \{s \in [0, t] \mid \text{there exists an element of } \Gamma_N(r) \text{ passing through } s\}$$

It is clear that there exists a point  $r_+$  (respectively  $r_-$ ) in  $W_I(y_0, y_1) \cap W_J(y_0, y_1)$  such that there exists an  $N$ -segment to  $r_+$  (respectively  $r_-$ ) passing through  $t_+$  (respectively  $t_-$ ). Suppose that  $r_+ \neq r_-$ . By applying Lemma 3.3, we get a cut point  $r \in W_I(r_-, r_+) \cap W_J(r_-, r_+)$  such that there exists an  $N$ -segment to  $r$  passing through a point in  $(t_-, t_+)$ .

Notice that  $t_- < t < t_+$ , since we assumed that there does not exist an  $N$ -segment to a point in  $W_I(y_0, y_1) \cap W_J(y_0, y_1)$  passing through the point  $t$ . This contradicts the definitions of  $t_+$  and  $t_-$ . Thus,  $r_+ = r_-$ , and there exists a sector at  $r_+ = r_- \in W_I(y_0, y_1) \cap W_J(y_0, y_1)$  containing  $t$ . The uniqueness of the existence of the point  $r$  is clear, since  $r \in W_I(y_0, y_1) \cap W_J(y_0, y_1) \subset C_N$ , and an  $N$ -segment does not intersect any other  $N$ -segment at its interior point.  $\square$

**Proposition 3.5** *Let  $x$  be a cut point of  $N$ , and  $B_{\delta_0}(x)$  a strongly convex neighborhood at  $x$ . Then, there exists  $\delta \in (0, \delta_0)$  such that any cut point  $y \in B_\delta(x) \cap C_N$  can be joined with  $x$  by a Jordan arc in  $B_{\delta_0}(x) \cap C_N$ .*

*Proof.* Choose a sufficiently small positive  $\delta$ , so that  $W_{I_z}(x, z) \cap W_{J_z}(x, z) \subset B_{\delta_0}(x)$  for any  $z \in B_\delta(x) \setminus \{x\}$ . Here  $I_z$  and  $J_z$  denote the open subarcs of  $S_{2\delta_0}(x)$  that form part of the boundary of  $W(x, z) := \Sigma_x(z) \cap \Sigma_z(x)$ , and  $\Sigma_z(x)$  (respectively  $\Sigma_x(z)$ ) denotes the sector at  $z$  (respectively at  $x$ ) containing  $x$  (respectively  $z$ ). Choose any  $y \in C_N \cap B_\delta(x) \setminus \{x\}$  and fix it. Since  $I$  and its closure  $\bar{I}$  are Jordan arcs, we may assume that  $I = (0, 1)$  and  $\bar{I} = [0, 1]$ . Here  $I$  and  $J$  denote the subarc of  $S_{2\delta_0}(x)$  corresponding to the cut point  $y$ . Here we assume that the  $N$ -segment to  $y$  (respectively  $x$ ) forming the boundary of  $W_I(x, y) \cap W_J(x, y)$  passing through the point 0 (respectively 1), which is an endpoint of  $I$ .

We will construct a homeomorphism from  $\bar{I}$  into  $C_N \cap B_{\delta_0}(x)$ . Choose any  $t \in I$  and fix it. If there exists a cut point  $z \in W_I(x, y) \cap W_J(x, y)$  such that a minimal geodesic segment in  $\Gamma_N(z)$  passes through  $t$ , we define  $\xi(t) = z$ . Suppose that there is no such a cut point  $z \in W_I(x, y) \cap W_J(x, y)$  for  $t$ . Then, from Lemma 3.4, it follows that there exists a sector  $\Sigma_r$  at  $r$  containing  $t$  for some cut point  $r \in W_I(y_0, y_1) \cap W_J(y_0, y_1)$ . We define  $\xi(t) = r$  for such a  $t$ . Hence we have constructed a continuous map  $\xi$  from  $\bar{I}$  into  $B_{\delta_0}(x) \cap C_N$ , where we define  $\xi(0) = y$  and  $\xi(1) = x$ .

It is clear that if  $\xi(t_1) = \xi(t_2)$  holds for distinct  $t_1, t_2 \in \bar{I}$ , then there exists an interval  $[a, b] \subset \bar{I}$  such that  $\xi|_{[a, b]} = \xi(t_1)$ ,  $t_1, t_2 \in [a, b]$ . Hence there exist countably many mutually disjoint subintervals  $\{I_n\}_n$  of  $\bar{I}$ , such that  $\xi(t_1) = \xi(t_2)$  holds for distinct  $t_1, t_2$  if and only if  $t_1$  and  $t_2$  are elements of a common  $I_n$ .

Let  $f : [0, 1] \rightarrow [0, 1]$  be a continuous non-decreasing function such that  $f(0) = 0$ ,  $f(1) = 1$  and such that  $f(t_1) = f(t_2)$  for distinct  $t_1, t_2$  if and only if  $t_1$  and  $t_2$  lie in a common  $I_n$  (the existence of the function  $f$  is proved in Lemma 4.1.3 in [SST]).

Then the curve  $c : [0, 1] \rightarrow B_{\delta_0}(x) \cap C_N$  defined by

$$c(u) := \xi(\max f^{-1}(u))$$

is injective and continuous. Hence, the cut points  $y$  and  $x$  can be joined by a Jordan arc in  $B_{\delta_0}(x) \cap C_N$ .  $\square$

A topological set  $T$  is called a *tree* if any two points in  $T$  can be joined by a unique Jordan arc in  $T$ . Likely, a topological set  $C$  is called a *local tree* if for every point  $x \in C$  and for any neighborhood  $U$  of  $x$ , there exists a neighborhood  $V \subset U$  of  $x$  such that  $C$  is a tree. A point of a local tree  $C$  is called an *endpoint* of the local tree if there exists a unique sector at  $x$ .

**Theorem 3.6** *Let  $N$  be a closed subset of a (forward) complete 2-dimensional Finsler manifold  $M$ . Then the cut locus of  $N$  is a local tree.*

*Proof.* Let  $x$  be a cut point of  $N$ , and  $U$  a neighborhood of  $x$ . Choose a strongly convex ball  $B_{4\delta_0}(x) \subset U$ . Let  $\delta$  be a positive number guaranteed in Proposition 3.5. Let  $\Sigma$  denote the intersection of all  $\Sigma_y(x)$ , where  $y \in S_\delta(x) \cap C_N$ . From Proposition 3.5, it follows that any point  $y \in \Sigma \cap C_N \cap B_\delta(x)$  can be joined by a Jordan arc  $c$  in  $B_{\delta_0}(x) \cap C_N$ . Since the curve  $c$  does not intersect  $S_\delta(x)$ , the curve lies in the set  $\Sigma \cap B_\delta(x) \subset U$ . Hence any cut point of  $N$  in  $\Sigma \cap B_\delta(x)$  can be joined to the point  $x$  by a Jordan arc in  $\Sigma \cap C_N \cap B_\delta(x)$ . This implies that any two points in  $\Sigma \cap C_N \cap B_\delta(x)$  can be joined by a Jordan arc in  $\Sigma \cap C_N \cap B_\delta(x)$  by way of  $x$ . Suppose that there exist two Jordan arcs in  $\Sigma \cap C_N \cap B_\delta(x)$  joining two cut points of  $N$  in  $\Sigma \cap B_\delta(x)$ . Then, the Jordan arcs contain a Jordan curve  $\alpha$  as a subset in the convex ball  $B_\delta(x)$ . Take a point  $z$  in the domain bounded by  $\alpha$ . Any  $N$ -segments to  $z$  intersect  $\alpha \subset C_N$ . This is a contradiction. Thus, any two points in  $\Sigma \cap C_N \cap B_\delta(x)$  is joined by a unique curve in the set. It is trivial that  $\Sigma \cap C_N \cap B_\delta(x)$  is a neighborhood of  $x$  since any  $N$ -segment to a point of  $C_N \cap S_\delta(x)$  does not pass through the point  $x$ . Therefore,  $\Sigma \cap C_N \cap B_\delta(x)$  is a tree and a neighborhood of  $x$  in  $C_N$ .  $\square$

## 4 Key lemmas

In this section, two key lemmas (Lemmas 4.3 and 4.4) are proved. Before stating them, we need two fundamental lemmas which are true for any dimensional Finsler manifolds. The first one is well known (see for example [BCS], Lemma 6.2.1).

**Lemma 4.1** *Let  $(M, F)$  be a (forward) complete Finsler manifold. Then, for each positive number  $a > 0$ , there exists a constant  $\lambda(a) > 1$  such that*

$$\lambda(a)^{-1}d(y, x) \leq d(x, y) \leq \lambda(a)d(y, x)$$

for any  $x, y \in B_a(p)$ .

**Lemma 4.2** *Let  $(M, F)$  be a (forward) complete Finsler manifold and, let  $\alpha : [0, \infty) \times [0, 2\pi] \rightarrow M$  denote the map defined by*

$$\alpha(t, \theta) := \exp_p(tv(\theta)),$$

where  $v(\theta)$  denotes a parametrization of the indicatrix curve  $S_p M = \{v \in T_p M \mid F(p, v) = 1\}$ , and  $\theta$  denotes the usual Euclidean angle.

Then for each  $a > 0$ , there exists a positive constant  $\mathcal{C}(a)$  such that

$$F\left(\frac{\partial \alpha}{\partial \theta}(t, \theta)\right) \leq \mathcal{C}(a),$$

for any  $t \in [0, a]$  and any  $\theta \in [0, 2\pi]$ .



*Proof.* For each  $\theta$ ,

$$Y_\theta(t) := \frac{\partial \alpha}{\partial \theta}(t, \theta)$$

is a Jacobi field along the geodesic  $\gamma_\theta(t) := \alpha(t, \theta)$  (see p. 130 in [BCS] or p.167 in [S] for the details on the Jacobi equation in Finsler geometry). The Jacobi field  $Y_\theta(t)$  satisfies the differential equation

$$D_T D_T Y_\theta(t) + R(Y_\theta(t), \dot{\gamma}_\theta(t)) \dot{\gamma}_\theta(t) = 0$$

with initial conditions

$$Y_\theta(0) = 0, \quad D_T Y_\theta(0) = \frac{\partial v}{\partial \theta}(0).$$

Here  $D_T$  denotes the absolute derivative along  $\gamma_\theta(t)$  with reference vector  $T(t) := \dot{\gamma}_\theta(t)$  and  $R$  denotes the  $h$ -curvature of  $M$ . Since  $Y_\theta(t)$  depends continuously on the initial conditions, there exists a constant  $\mathcal{C}(a)$  such that

$$F(Y_\theta(t)) \leq \mathcal{C}(a)$$

for any  $t \in [0, a]$  and  $\theta \in [0, 2\pi]$ . □

We define the length  $l(c)$  of a continuous curve  $c : [a, b] \rightarrow M$  by

$$l(c) := \sup \left\{ \sum_{i=1}^k d(c(t_{i-1}), c(t_i)) \mid a =: t_0 < t_1 < \cdots < t_{k-1} < t_k := b \right\}. \quad (4.1)$$

From now on, we will fix a Jordan arc  $c : [0, 1] \rightarrow C_N$  in the cut locus of a closed subset  $N$  of a (forward) complete 2-dimensional Finsler manifold  $(M, F)$ .

**Lemma 4.3** *Let  $[a, b]$  be a subinterval of  $[0, 1]$ . Suppose that there exists a positive number  $\epsilon_0$  such that for each  $t \in [a, b)$  (respectively  $t \in (a, b]$ ),*

$$\lim_{t \searrow t_0} D_N(c(t_0), c(t)) > \epsilon_0$$

$$\text{(respectively } \lim_{t \nearrow t_0} D_N(c(t), c(t_0)) > \epsilon_0 \text{)}.$$

*Then, the length  $l(c)$  of  $c$  is not greater than  $\frac{1}{\epsilon_0}(d(N, c(b)) - d(N, c(a)))$ , i.e.,  $c$  is rectifiable. Here,*

$$D_N(x, y) := \frac{d(N, y) - d(N, x)}{d(x, y)}.$$

*Proof.* From our assumption, for any sufficiently fine subdivision  $u_0 := a < u_1 < \cdots < u_{n-1} < u_n := b$  of  $[a, b]$ ,

$$d(N, c(u_{i+1})) - d(N, c(u_i)) > \epsilon_0 d(c(u_i), c(u_{i+1}))$$

holds for each  $i = 0, 1, \dots, n-1$ . Therefore, the length of  $c$  is not greater than

$$\frac{1}{\epsilon_0} \sum_{i=0}^{n-1} (d(N, c(u_{i+1})) - d(N, c(u_i))) = \frac{1}{\epsilon_0} (d(N, c(b)) - d(N, c(a))).$$

□



**Lemma 4.4** *Let  $[a, b]$  be a subinterval of  $[0, 1]$ . Suppose that there exists a point  $p \in M$  such that for each  $t \in [a, b]$ , the minimal geodesic segment from  $p$  to  $c(t)$  does not intersect  $c[a, b]$  except  $c(t)$  and such that  $c[a, b]$  is disjoint from the cut locus of  $p$  and  $P := \{p\}$ . Suppose that there exists  $\epsilon_0 \in (0, 1)$  such that*

$$\lim_{t \searrow t_0} D_P(c(t_0), c(t)) < \epsilon_0, \quad (4.2)$$

$$(\text{respectively } \lim_{t \nearrow t_0} D_P(c(t), c(t_0)) < \epsilon_0) \quad (4.3)$$

$$\lim_{t \searrow t_0} D_P(c(t), c(t_0)) < \epsilon_0 \quad (4.4)$$

$$(\text{respectively } \lim_{t \nearrow t_0} D_P(c(t_0), c(t)) < \epsilon_0) \quad (4.5)$$

for each  $t_0 \in [a, b]$  (respectively  $t_0 \in (a, b]$ .) Here,

$$D_P(x, y) := \frac{d(p, y) - d(p, x)}{d(x, y)}.$$

Then the curve  $c$  is rectifiable on  $[a, b]$ .

*Proof.* Let  $v(\theta)$  denote a curve emanating from  $v_0 := \frac{1}{F(\exp_p^{-1}(c(a)))} \exp_p^{-1}(c(a))$  in  $S_p M$ . Here the parameter  $\theta$  denotes the oriented Euclidean angle measured from  $v_0$  to  $v(\theta)$ . By the assumption of our lemma, the curve  $c$  is parametrized by  $\theta$  ;

$$m(\theta) = \exp_p(\rho(\theta)v(\theta)), \quad \theta \in [0, \theta_0].$$

Here  $\rho(\theta) = F(\exp_p^{-1}(c(t)))$ ,  $v(\theta) = \frac{1}{\rho(\theta)} \exp_p^{-1}(c(t))$ ,  $m(0) = c(a)$ , and  $m(\theta_0) = c(b)$ . From (4.2), (4.3), (4.4) and (4.5), it follows that for any sufficiently fine subdivision  $u_0 := 0 < u_1 < u_2 < \dots < u_n := \theta_0$  of  $[0, \theta_0]$ ,

$$D_P(m(u_i), m(u_{i+1})) < \epsilon_0 \quad (4.6)$$

and

$$D_P(m(u_{i+1}), m(u_i)) < \epsilon_0 \quad (4.7)$$

hold for each  $i = 0, 1, 2, \dots, n-1$ .

Suppose first that

$$l_i := d(p, m(u_i)) \leq l_{i+1} := d(p, m(u_{i+1}))$$

for some fixed  $i$ . By the triangle inequality,

$$d(m(u_i), m(u_{i+1})) \leq d(m(u_i), \gamma_{i+1}(l_i)) + l_{i+1} - l_i. \quad (4.8)$$

Here  $\gamma_{i+1} : [0, l_{i+1}] \rightarrow M$  denotes the geodesic  $\exp_p(tv(u_{i+1}))$ .

By applying Lemma 4.2 to the curve  $\{\exp_p(l_i v(\theta)) | u_i \leq \theta \leq u_{i+1}\}$ , we get, by Lemma 4.3,

$$d(m(u_i), \gamma_{i+1}(l_i)) \leq \mathcal{C}(\bar{a})(u_{i+1} - u_i), \quad (4.9)$$

where  $\bar{a} := \max\{d(p, c(t)) \mid a \leq t \leq b\}$ . Combining (4.6), (4.8) and (4.9) we obtain

$$d(m(u_i), m(u_{i+1})) \leq \frac{\mathcal{C}(\bar{a})}{1 - \epsilon_0}(u_{i+1} - u_i) \quad (4.10)$$

for any  $i$  with  $d(p, m(u_i)) \leq d(p, m(u_{i+1}))$ .

Suppose second that

$$l_i = d(p, m(u_i)) > l_{i+1} = d(p, m(u_{i+1}))$$

for some fixed  $i$ . Then, by a similar argument as above, we get

$$d(m(u_{i+1}), m(u_i)) \leq \mathcal{C}(\bar{a})(u_{i+1} - u_i) + l_i - l_{i+1}. \quad (4.11)$$

Combining (4.7) and (4.11), we obtain

$$d(m(u_{i+1}), m(u_i)) \leq \frac{\mathcal{C}(\bar{a})}{1 - \epsilon_0}(u_{i+1} - u_i).$$

Hence, by Lemma 4.1,

$$d(m(u_i), m(u_{i+1})) \leq \frac{\lambda(\bar{a})}{1 - \epsilon_0} \mathcal{C}(\bar{a})(u_{i+1} - u_i)$$

for any  $i$  with  $d(p, m(u_i)) > d(p, m(u_{i+1}))$ . Therefore, the length  $l(c)$  of  $c$  does not exceed

$$\frac{\lambda(\bar{a})}{1 - \epsilon_0} \mathcal{C}(\bar{a})\theta_0, \quad (4.12)$$

i.e., the curve  $c$  is rectifiable.  $\square$

## 5 Fundamental properties of a Jordan arc in the cut locus

Let us recall that  $c : [0, 1] \rightarrow C_N$  is a Jordan arc in the cut locus of the closed subset  $N$ . For each  $t \in [0, 1]$  (respectively  $t \in (0, 1]$ ), let  $\Sigma_{c(t)}^+$  (respectively  $\Sigma_{c(t)}^-$ ) denote the sector at  $c(t)$  that contains  $c(t, t + \delta)$  (respectively  $c(t - \delta, t)$ ) for some small  $\delta > 0$ . Let  $\alpha_t^+$  and  $\beta_t^+$  (respectively  $\alpha_t^-$  and  $\beta_t^-$ ) denote the unit speed  $N$ -segments to  $c(t)$  that form part of the boundary of  $\Sigma_{c(t)}^+$  (respectively  $\Sigma_{c(t)}^-$ ). Notice that for each  $t \in (0, 1)$ ,  $\alpha_t^+ \neq \beta_t^+$ , and  $\alpha_t^- \neq \beta_t^-$ .

Then, with the notations above, we have the following important result.

**Proposition 5.1** *Suppose that  $\alpha_t^+ \neq \beta_t^+$  for  $t = 0$ . Then for each  $t_0 \in [0, 1]$ , the following limits from the right exist:*

$$v^f(t_0)^+ := \lim_{t \searrow t_0} \frac{1}{F(\exp_{c(t_0)}^{-1} c(t))} \exp_{c(t_0)}^{-1}(c(t)) \quad (5.1)$$

and

$$v^b(t_0)^+ := \lim_{t \searrow t_0} \frac{1}{F(\exp_{c(t)}^{-1} c(t_0))} \exp_{c(t)}^{-1}(c(t_0)). \quad (5.2)$$

*Proof.* Since  $S_{c(t_0)}M := \{v \in T_{c(t_0)}M \mid F(v) = 1\}$  is compact, any sequence

$$\left\{ \frac{1}{F(\exp_{c(t_0)}^{-1} c(t_0 + \epsilon_i))} \exp_{c(t_0)}^{-1} c(t_0 + \epsilon_i) \right\}$$

has a limit, where  $\{\epsilon_i\}$  denotes a sequence of positive numbers convergent to zero. Let  $v_0 \in S_{c(t_0)}M$  be a limit of the sequence above. By choosing a subsequence, we may assume that the sequence has a unique limit. From Proposition 2.2, it follows that

$$\lim_{i \rightarrow \infty} \frac{d(N, c(t_0 + \epsilon_i)) - d(N, c(t_0))}{d(c(t_0), c(t_0 + \epsilon_i))} = g_X(X, v_0) = g_Y(Y, v_0),$$

where  $X$  and  $Y$  are the tangent vectors of the unit speed  $N$ -segment  $\alpha_{t_0}^+$  and  $\beta_{t_0}^+$  at  $c(t_0)$ , respectively. Since  $X \neq Y$ , the space  $\{Z \in T_{c(t_0)}M \mid g_X(X, Z) = g_Y(Y, Z)\}$  is a 1-dimensional linear subspace of  $T_{c(t_0)}M$ . On the other hand, it is clear that any limits of  $\frac{1}{F(\exp_{c(t_0)}^{-1} c(t))} \exp_{c(t_0)}^{-1} c(t)$  as  $t \searrow t_0$  lie in the common subarc  $J^+(X, Y)$  of  $S_{c(t_0)}M$  with endpoints  $X, Y$ . Hence, the limit  $v_0$  is the unique element of  $J^+(X, Y) \cap \{Z \in T_{c(t_0)}M \mid g_X(X, Z) = g_Y(Y, Z)\}$ . This implies that the limit (5.1) exists. By applying Proposition 2.3, we can easily see that the limit (5.2) exists.  $\square$

By reversing the parameter of  $c$  in Proposition 5.1, we have the following proposition.

**Proposition 5.2** *Suppose that  $\alpha_t^+ \neq \beta_t^+$  for  $t = 1$ . Then for each  $t_0 \in (0, 1]$ , the following limits from the left exist:*

$$v^f(t_0)^- := \lim_{t \nearrow t_0} \frac{1}{F(\exp_{c(t_0)}^{-1} c(t))} \exp_{c(t_0)}^{-1} c(t)$$

and

$$v^b(t_0)^- := \lim_{t \nearrow t_0} \frac{1}{F(\exp_{c(t)}^{-1} c(t_0))} \exp_{c(t)}^{-1} c(t_0).$$

**Lemma 5.3** *If  $\Gamma_N(c(0))$  consists of a unique element  $\alpha$ , then*

$$\lim_{t \searrow 0} v^f(t)^+ = \lim_{t \searrow 0} v^b(t)^- = \dot{\alpha}(l),$$

where  $l = d(N, c(0))$ .

*Proof.* From the proof of Proposition 5.1,  $v^f(t)^+$  (respectively  $v^b(t)^-$ ) is the unique element of the set  $J^+(X_t, Y_t) \cap \{Z \in S_{c(t)}M \mid g_{X_t}(X_t, Z) = g_{Y_t}(Y_t, Z)\}$ . Here  $X_t = \dot{\alpha}_t^+(d(N, c(t)))$  (respectively  $X_t = \dot{\alpha}_t^-(d(N, c(t)))$ ) and  $Y_t = \dot{\beta}_t^+(d(N, c(t)))$  (respectively  $Y_t = \dot{\beta}_t^-(d(N, c(t)))$ ). Since  $\alpha$  is the unique element of  $\Gamma_N(c(0))$ , the arc  $J^+(X_t, Y_t)$  shrinks to  $\dot{\alpha}(l)$  as  $t \searrow 0$ . Therefore,  $\lim_{t \searrow 0} v^f(t)^+ = \lim_{t \searrow 0} v^b(t)^- = \dot{\alpha}(l)$ .  $\square$

**Lemma 5.4** Suppose that  $\alpha_t^+ \neq \beta_t^+$  for  $t = 0$ . Then for each  $t_0 \in [0, 1)$

$$\lim_{t \searrow t_0} v^f(t)^+ = v^f(t_0)^+, \quad \text{and} \quad \lim_{t \searrow t_0} v^f(t)^- = \frac{-1}{F(-v^f(t_0)^+)} v^f(t_0)^+.$$

*Proof.* From the proof of Proposition 5.1,  $v^f(t)^*$ , where  $*$  denotes  $+$  or  $-$ , is the unique element of the set

$$J^*(X_t, Y_t) \cap \{Z \in T_{c(t)}M \mid g_{X_t}(X_t, Z) = g_{Y_t}(Y_t, Z)\}$$

for each  $t \in (0, 1)$ . Here  $X_t = \dot{\alpha}_t^+(d(N, c(t)))$ , and  $Y_t = \dot{\beta}_t^+(d(N, c(t)))$ , and  $J^-(X_t, Y_t)$  denotes the complementary subarc of  $J^+(X_t, Y_t)$  in  $S_{c(t_0)}M$ . Since  $\lim_{t \searrow t_0} \alpha_t^* = \alpha_{t_0}^+$  and  $\lim_{t \searrow t_0} \beta_t^* = \beta_{t_0}^+$ , we have  $\lim_{t \searrow t_0} X_t = X_{t_0}$ , and  $\lim_{t \searrow t_0} Y_t = Y_{t_0}$ . This implies that  $\lim_{t \searrow t_0} v^f(t)^+ = v^f(t_0)^+$  and  $\lim_{t \searrow t_0} v^f(t)^- = \frac{-1}{F(-v^f(t_0)^+)} v^f(t_0)^+$ .  $\square$

**Lemma 5.5** Suppose that  $\alpha_t^+ \neq \beta_t^+$  for  $t = 0$ . Then for each  $t_0 \in [0, 1)$ , the following limits from the right exist:

$$\lim_{t \searrow t_0} D_N(c(t_0), c(t)) = g_{w(t_0)^+}(w(t_0)^+, v^f(t_0)^+) \quad (5.3)$$

and

$$\lim_{t \searrow t_0} D_N(c(t), c(t_0)) = g_{w(t_0)^+}(w(t_0)^+, v^b(t_0)^+), \quad (5.4)$$

where

$$w(t_0)^+ := \dot{\alpha}_{t_0}^+(d(N, c(t_0))) \quad \text{or} \quad \dot{\beta}_{t_0}^+(d(N, c(t_0))).$$

Furthermore, for each compact subinterval  $[a, b] \subset [0, 1)$ , there exists  $\epsilon_0 \in (0, 1)$  such that

$$\lim_{t \searrow t_0} D_N(c(t_0), c(t)) < \epsilon_0 \quad \text{and} \quad \lim_{t \searrow t_0} D_N(c(t), c(t_0)) < \epsilon_0$$

for each  $t_0 \in [a, b]$ .

*Proof.* From Propositions 2.2, 2.3, and 5.1 it follows that for each  $t_0 \in [0, 1)$

$$N_+^f(t_0) := \lim_{t \searrow t_0} D_N(c(t_0), c(t)) = g_X(X, v^f(t_0)^+) = g_Y(Y, v^f(t_0)^+) \quad (5.5)$$

and

$$N_+^b(t_0) := \lim_{t \searrow t_0} D_N(c(t), c(t_0)) = g_X(X, v^b(t_0)^+) = g_Y(Y, v^b(t_0)^+) \quad (5.6)$$

hold. Here  $X := \dot{\alpha}_{t_0}^+(d(N, c(t_0)))$  and  $Y := \dot{\beta}_{t_0}^+(d(N, c(t_0)))$ . Hence (5.3) and (5.4) are clear. We will prove the latter claim. From Lemma 1.2.3 in [S], it is clear that  $N_+^f(t_0) < 1$  and  $N_+^b(t_0) < 1$  for each  $t_0 \in [0, 1)$ . Notice that  $X \neq Y$ , since  $\alpha_{t_0}^+ \neq \beta_{t_0}^+$ . Suppose that there exists a sequence  $\{s_j\}$  of numbers in  $[a, b]$  satisfying

$$\lim_{j \rightarrow \infty} N_+^f(s_j) = 1 \quad \text{or} \quad \lim_{j \rightarrow \infty} N_+^b(s_j) = 1 \quad (5.7)$$

Thus, by (5.5), (5.6) and (5.7),

$$\lim_{j \rightarrow \infty} g_{X_j}(X_j, v^f(s_j)^+) = \lim_{j \rightarrow \infty} g_{Y_j}(Y_j, v^f(s_j)^+) = 1 \quad (5.8)$$

or

$$\lim_{j \rightarrow \infty} g_{X_j}(X_j, v^b(s_j)^+) = \lim_{j \rightarrow \infty} g_{Y_j}(Y_j, v^b(s_j)^+) = 1 \quad (5.9)$$

holds. Here,  $X_j := \dot{\alpha}_{s_j}^+(d(N, c(s_j)))$  and  $Y_j := \dot{\beta}_{s_j}^+(d(N, c(s_j)))$ .

By choosing a subsequence of  $\{s_j\}$ , we may assume that  $s_\infty := \lim_{j \rightarrow \infty} s_j \in [a, b]$ ,  $\alpha_\infty := \lim_{j \rightarrow \infty} \alpha_j^+$ ,  $\beta_\infty := \lim_{j \rightarrow \infty} \beta_j^+$ ,  $X_\infty := \lim_{j \rightarrow \infty} X_j = \dot{\alpha}_\infty(d(N, c(s_\infty)))$ ,  $Y_\infty := \lim_{j \rightarrow \infty} Y_j = \dot{\beta}_\infty(d(N, c(s_\infty)))$ ,  $\lim_{j \rightarrow \infty} v^f(s_j)^+$  and  $\lim_{j \rightarrow \infty} v^b(s_j)^+ (\in T_{c(s_\infty)}M)$  exist. By (5.8) and (5.9), we obtain

$$g_{X_\infty}(X_\infty, \lim_{j \rightarrow \infty} v^f(s_j)^+) = g_{Y_\infty}(Y_\infty, \lim_{j \rightarrow \infty} v^f(s_j)^+) = 1 \quad (5.10)$$

or

$$g_{X_\infty}(X_\infty, \lim_{j \rightarrow \infty} v^b(s_j)^+) = g_{Y_\infty}(Y_\infty, \lim_{j \rightarrow \infty} v^b(s_j)^+) = 1 \quad (5.11)$$

holds.

Since  $\alpha_\infty$  and  $\beta_\infty$  are  $N$ -segments that form part of the sector  $\Sigma_{c(s_\infty)}^+$  or  $\Sigma_{c(s_\infty)}^-$  at  $c(s_\infty)$ ,  $s_\infty \in [a, b] \subset [0, 1)$ , it follows from our assumption that  $X_\infty \neq Y_\infty$ . Thus, by Lemma 1.2.3 in [S], we get a contradiction from the equations (5.10) and (5.11). This implies the existence of the number  $\epsilon_0 \in (0, 1)$ .  $\square$

Similarly, we have

**Lemma 5.6** *Suppose that  $\alpha_t^+ \neq \beta_t^+$  for  $t = 0$ . Then for each  $t_0 \in (0, 1)$ , the following limits from the left exist:*

$$\lim_{t \nearrow t_0} D_N(c(t_0), c(t)) = g_{w(t_0)^-}(w(t_0)^-, v^f(t_0)^-) \quad (5.12)$$

and

$$\lim_{t \nearrow t_0} D_N(c(t), c(t_0)) = g_{w(t_0)^-}(w(t_0)^-, v^b(t_0)^-), \quad (5.13)$$

where

$$w(t_0)^- := \dot{\alpha}_{t_0}^-(d(N, c(t_0))) \text{ or } \dot{\beta}_{t_0}^-(d(N, c(t_0))).$$

Furthermore, for each compact subinterval  $[a, b] \subset [0, 1)$ , there exists  $\epsilon_0 \in (0, 1)$  such that

$$\lim_{t \nearrow t_0} D_N(c(t_0), c(t)) < \epsilon_0, \quad \text{and} \quad \lim_{t \nearrow t_0} D_N(c(t), c(t_0)) < \epsilon_0$$

for each  $t_0 \in (a, b]$ .

## 6 Approximation by the distance function from a point

**Lemma 6.1** *Suppose that  $\alpha_t^+ \neq \beta_t^+$  for  $t = 0$ . Then for any  $t_0 \in [0, 1)$  and each interior point  $p$  of the  $N$ -segment  $\alpha_{t_0}^+$ , there exist positive numbers  $\epsilon_1 \in (0, 1)$  and  $\delta_0$  such that*

$$\lim_{u \searrow t} D_P(c(t), c(u)) < \epsilon_1 \quad \text{and} \quad \lim_{u \searrow t} D_P(c(u), c(t)) < \epsilon_1$$

for each  $t \in [t_0, t_0 + \delta_0)$ . Here  $P := \{p\}$ .

*Proof.* Suppose that  $t_0 \in [0, 1)$  is arbitrarily given. Choose any interior point  $p$  of the  $N$ -segment  $\alpha_{t_0}^+$ . Since the point  $c(t_0)$  is not a cut point of the point  $p$ , there exists  $\delta_1 \in (0, 1 - t_0)$  such that the subarc  $c[t_0, t_0 + \delta_1]$  of  $c$  is disjoint from the cut locus of  $p$ . Notice that the cut locus of  $p$  is a closed subset of  $M$ . By applying Lemma 5.5 for the interval  $[t_0, t_0 + \delta_1]$ , we get a number  $\epsilon_0 \in (0, 1)$  satisfying

$$\lim_{u \searrow t} D_N(c(t), c(u)) = g_{w(t)^+}(w(t)^+, v^f(t)^+) < \epsilon_0, \quad (6.1)$$

and

$$\lim_{u \searrow t} D_N(c(u), c(t)) = g_{w(t)^+}(w(t)^+, v^b(t)^+) < \epsilon_0 \quad (6.2)$$

for each  $t \in [t_0, t_0 + \delta_1]$ . Here  $w(t)^+$  denotes  $\dot{\alpha}_t^+(d(N, c(t)))$  in our argument.

For each  $t \in [t_0, t_0 + \delta_1]$ , let  $(\nabla d_p)_{c(t)}$  denote the (unit) velocity vector of the minimal geodesic segment from  $p$  to  $c(t)$  at  $c(t)$ . Since  $(\nabla d_p)_{c(t_0)} = w(t_0)^+$  and  $\lim_{t \searrow t_0} w(t)^+ = w(t_0)^+$ , we get a number  $\delta_0 \in (0, \delta_1)$  so as to satisfy that  $F((\nabla d_p)_{c(t)} - w(t)^+)$  is sufficiently small for each  $t \in [t_0, t_0 + \delta_0]$ , so that

$$\left| g_{(\nabla d_p)_{c(t)}}((\nabla d_p)_{c(t)}, v^f(t)^+) - g_{w(t)^+}(w(t)^+, v^f(t)^+) \right| < \frac{1 - \epsilon_0}{2} \quad (6.3)$$

and

$$\left| g_{(\nabla d_p)_{c(t)}}((\nabla d_p)_{c(t)}, v^b(t)^+) - g_{w(t)^+}(w(t)^+, v^b(t)^+) \right| < \frac{1 - \epsilon_0}{2} \quad (6.4)$$

hold for each  $t \in [t_0, t_0 + \delta_0]$ . Therefore, by the triangle inequality and the equations (6.1), (6.2), (6.3) and (6.4),

$$g_{(\nabla d_p)_{c(t)}}((\nabla d_p)_{c(t)}, v^f(t)^+) < \frac{1 + \epsilon_0}{2} \quad (6.5)$$

and

$$g_{(\nabla d_p)_{c(t)}}((\nabla d_p)_{c(t)}, v^b(t)^+) < \frac{1 + \epsilon_0}{2} \quad (6.6)$$

for each  $t \in [t_0, t_0 + \delta_0]$ .

On the other hand, by Propositions 2.2, 2.3 and 5.1, we obtain

$$\lim_{u \searrow t} D_P(c(t), c(u)) = g_{(\nabla d_p)_{c(t)}}((\nabla d_p)_{c(t)}, v^f(t)^+) \quad (6.7)$$

$$\lim_{u \searrow t} D_P(c(u), c(t)) = g_{(\nabla d_p)_{c(t)}}((\nabla d_p)_{c(t)}, v^b(t)^+), \quad (6.8)$$

for each  $t \in [t_0, t_0 + \delta_0]$ . From (6.5), (6.6), (6.7) and (6.8), it is clear that  $\lim_{u \searrow t} D_P(c(t), c(u))$  and  $\lim_{u \searrow t} D_P(c(u), c(t))$  are less than  $\epsilon_1 := \frac{1 + \epsilon_0}{2}$  for each  $t \in [t_0, t_0 + \delta_0]$ .  $\square$

**Lemma 6.2** *Suppose that  $\alpha_t^+ \neq \beta_t^+$  for  $t = 0$ . Then for any  $t_0 \in [0, 1)$  and each interior point  $p$  of the  $N$ -segment  $\alpha_{t_0}^+$ , there exist positive numbers  $\epsilon_1 \in (0, 1)$  and  $\delta_0$  such that*

$$\lim_{u \nearrow t} D_P(c(t), c(u)) < \epsilon_1 \quad \text{and} \quad \lim_{u \nearrow t} D_P(c(u), c(t)) < \epsilon_1$$

for each  $t \in (t_0, t_0 + \delta_0]$ .

*Proof.* Suppose that  $t_0 \in [0, 1)$  is arbitrarily given. Choose any interior point  $p$  of the  $N$ -segment  $\alpha_{t_0}^+$ . Since the point  $c(t_0)$  is not a cut point of the point  $p$ , there exists  $\delta_1 \in (0, 1 - t_0)$  such that the subarc  $c[t_0, t_0 + \delta_1]$  of  $c$  is disjoint from the cut locus of  $p$ . By applying Lemma 5.6 for the interval  $[t_0, t_0 + \delta_1]$ , we get a number  $\epsilon_0 \in (0, 1)$  satisfying

$$\lim_{u \nearrow t} D_N(c(t), c(u)) = g_{w(t)^-}(w(t)^-, v^f(t)^-) < \epsilon_0, \quad (6.9)$$

and

$$\lim_{u \nearrow t} D_N(c(u), c(t)) = g_{w(t)^-}(w(t)^-, v^b(t)^-) < \epsilon_0 \quad (6.10)$$

for each  $t \in (t_0, t_0 + \delta_1]$ . Here  $w(t)^-$  denotes  $\dot{\alpha}_t^-(d(N, c(t)))$  in our argument. Since  $(\nabla d_p)_{c(t_0)} = w(t_0)^+$  and  $\lim_{t \searrow t_0} w(t)^- = w(t_0)^+$ , we get a number  $\delta_0 \in (0, \delta_1)$  so as to satisfy that  $F((\nabla d_p)_{c(t)} - w(t)^-)$  is sufficiently small for each  $t \in [t_0, t_0 + \delta_0]$ , so that

$$\left| g_{(\nabla d_p)_{c(t)}}((\nabla d_p)_{c(t)}, v^f(t)^-) - g_{w(t)^-}(w(t)^-, v^f(t)^-) \right| < \frac{1 - \epsilon_0}{2} \quad (6.11)$$

and

$$\left| g_{(\nabla d_p)_{c(t)}}((\nabla d_p)_{c(t)}, v^b(t)^-) - g_{w(t)^-}(w(t)^-, v^b(t)^-) \right| < \frac{1 - \epsilon_0}{2} \quad (6.12)$$

hold for each  $t \in (t_0, t_0 + \delta_0]$ . Therefore, by the triangle inequality and the equations (6.9), (6.10), (6.11) and (6.12),

$$g_{(\nabla d_p)_{c(t)}}((\nabla d_p)_{c(t)}, v^f(t)^-) < \frac{1 + \epsilon_0}{2} \quad (6.13)$$

and

$$g_{(\nabla d_p)_{c(t)}}((\nabla d_p)_{c(t)}, v^b(t)^-) < \frac{1 + \epsilon_0}{2} \quad (6.14)$$

for each  $t \in (t_0, t_0 + \delta_0]$ . On the other hand, by Propositions 2.2, 2.3 and 5.2, we obtain

$$\lim_{u \nearrow t} D_P(c(t), c(u)) = g_{(\nabla d_p)_{c(t)}}((\nabla d_p)_{c(t)}, v^f(t)^-) \quad (6.15)$$

and

$$\lim_{u \nearrow t} D_P(c(u), c(t)) = g_{(\nabla d_p)_{c(t)}}((\nabla d_p)_{c(t)}, v^b(t)^-) \quad (6.16)$$

for each  $t \in (t_0, t_0 + \delta_0]$ . From (6.13), (6.14), (6.15) and (6.16), it is clear that  $\lim_{u \nearrow t} D_P(c(t), c(u))$  and  $\lim_{u \nearrow t} D_P(c(u), c(t))$  are less than  $\epsilon_1 := \frac{1 + \epsilon_0}{2}$  for each  $t \in (t_0, t_0 + \delta_0]$ .  $\square$

**Lemma 6.3** *If  $c(0)$  admits a unique  $N$ -segment, then there exist  $\epsilon_0, \delta_0 \in (0, 1)$  satisfying that  $\lim_{u \searrow t} D_N(c(t), c(u)) > \epsilon_0$  for each  $t \in [0, \delta_0)$ , and  $\lim_{u \nearrow t} D_N(c(u), c(t)) > \epsilon_0$  for each  $t \in (0, \delta_0]$ .*

*Proof.* Since  $c(0)$  admits a unique  $N$ -segment  $\alpha : [0, l] \rightarrow M$ ,  $\lim_{t \searrow 0} w(t)^\pm = \dot{\alpha}(l)$ . Hence, by Lemmas 5.3, we obtain

$$\lim_{t \searrow 0} g_{w(t)^+}(w(t)^+, v^f(t)^+) = 1, \quad \text{and} \quad \lim_{t \searrow 0} g_{w(t)^-}(w(t)^-, v^b(t)^-) = 1.$$

Therefore, by (5.3) and (5.13), the existence of the numbers  $\epsilon_0$  and  $\delta_0$  is clear.  $\square$



**Theorem 6.4** *Any Jordan arc in the cut locus of a closed subset in a (forward) complete 2-dimensional Finsler manifold is rectifiable.*

*Proof.* Let  $c : [0, 1] \rightarrow C_N$  be a Jordan arc on the cut locus  $C_N$  of a closed subset  $N$ . Let  $t_0 \in [0, 1]$  be arbitrarily given. Suppose that  $\alpha_t^+ \neq \beta_t^+$  for  $t = t_0$ . Choose any interior point  $p$  of the  $N$ -segment  $\alpha_{t_0}^+$ . Then, from Lemmas 5.4, 6.1 and 6.2, it is clear that the point  $p$  satisfies the hypothesis of Lemma 4.4 for some interval  $[t_0, t_0 + \delta_0]$ . Therefore, from Lemma 4.4 it follows that  $c|_{[t_0, t_0 + \delta_0]}$  is rectifiable. Suppose next that  $\alpha_t^+ = \beta_t^+$  for  $t = t_0$ . This means that  $t_0 = 0$  and  $c(0)$  admits a unique  $N$ -segment. From Lemmas 4.3 and 6.3,  $c$  is rectifiable on  $[0, \delta_2] = [t_0, t_0 + \delta_2]$  for some positive  $\delta_2$ . By reversing the parameter of  $c$ , we have proved that for each  $t_0 \in [0, 1]$ , there exists an open interval containing  $t_0$  where  $c$  is rectifiable. This implies that  $c$  is rectifiable on  $[0, 1]$ .  $\square$

## 7 The topology induced by the intrinsic metric on the cut locus

Let  $N$  be a closed subset of a 2-dimensional (forward) complete Finsler manifold  $M$ . By Theorems 3.6 and 6.4, any two cut points  $y_1, y_2 \in C_N$  can be joined by a rectifiable arc in the cut locus  $C_N$  of  $N$  if  $y_1$  and  $y_2$  are in the same connected component of the cut locus of the closed subset  $N$ . Therefore, we can define the *intrinsic metric*  $\delta$  on  $C_N$  as follows:

1. if  $y_1, y_2 \in C_N$  are in the same connected component,

$$\delta(y_1, y_2) := \inf\{l(c) \mid c \text{ is a rectifiable arc in } C_N \text{ joining } y_1 \text{ and } y_2\},$$

2. otherwise  $\delta(y_1, y_2) := +\infty$ .

Recall that  $l(c)$ , given in (4.1), denotes the length for a continuous curve  $c : [a, b] \rightarrow M$ . It is fundamental that  $l(c)$  equals the integral length  $L(c)$  defined in (1.1) for any piecewise  $C^1$ -curve  $c$  (see [BM] for this proof). We will prove in Theorem 7.5 that  $l(c) = L(c)$  holds for any Lipschitz continuous curve  $c$ . Notice that for any Lipschitz curve  $c$ ,  $c(t)$  is differentiable for almost all  $t$ .

The following theorem for a Lipschitz function is a key tool in the argument below. The proof is immediate by taking into account Theorem 7.29 in [WZ], for example.

**Theorem 7.1** *If  $f : [a, b] \rightarrow \mathbb{R}$  is a Lipschitz function, its derivative function  $f'(t)$  exists for almost all  $t$  and*

$$f(t) = f(a) + \int_a^t f'(t) dt$$

*holds for any  $t \in [a, b]$ .*

**Lemma 7.2** *Let  $\gamma : [0, 1] \rightarrow M$  be a Lipschitz curve on  $M$  and let  $f : [0, 1] \rightarrow [0, \infty)$  be the distance function  $f(t) := d(q, \gamma(t))$  from a point  $q$ . Then  $f$  is a Lipschitz function and  $f'(t) \leq F(\dot{\gamma}(t))$  holds for almost all  $t \in (0, 1)$ .*

*Proof.* It follows from the triangle inequality that the function  $f$  is Lipschitz.

From Theorem 7.1 it follows that the curve  $\gamma$  and function  $f$  are differentiable almost everywhere. Using again the triangle inequality, for any small positive number  $h$ , we have

$$f(t+h) = d(q, \gamma(t+h)) \leq f(t) + d(\gamma(t), \gamma(t+h))$$

and therefore, if  $f'(t)$  exists, then

$$f'(t) = \lim_{h \searrow 0} \frac{f(t+h) - f(t)}{h} \leq \lim_{h \searrow 0} \frac{d(\gamma(t), \gamma(t+h))}{h} = F(\dot{\gamma}(t)),$$

where we have used the well-known Busemann-Mayer formula (see the original paper [BM], or a more modern treatment [BCS], p. 161). See also the proof of Lemma 7.6.  $\square$

**Lemma 7.3** *The length  $l(\gamma)$  of the Lipschitz curve  $\gamma$  satisfies*

$$l(\gamma|_{[0,a]}) + l(\gamma|_{[a,a+h]}) = l(\gamma|_{[0,a+h]}),$$

*for any non-negative  $h \leq 1 - a$ . In particular, the function  $l(c|_{[0,t]})$  is Lipschitz.*

*Proof.* It follows directly from (4.1).  $\square$

**Lemma 7.4** *For almost all  $t \in (0, 1)$  we have*

$$\frac{d}{dt} l(\gamma|_{[0,t]}) \geq F(\dot{\gamma}(t)).$$

*Proof.* Suppose that the function  $l(\gamma|_{[0,t]})$  and  $\gamma$  are differentiable at  $t = t_0$ . Using Lemma 7.3, we have

$$\left( \frac{d}{dt} \right)_{t_0} l(\gamma|_{[0,t]}) = \lim_{h \searrow 0} \frac{l(\gamma|_{[0,t_0+h]}) - l(\gamma|_{[0,t_0]})}{h} = \lim_{h \searrow 0} \frac{l(\gamma|_{[t_0,t_0+h]})}{h},$$

and from (4.1) it follows that

$$\left( \frac{d}{dt} \right)_{t_0} l(\gamma|_{[0,t_0]}) \geq \lim_{h \searrow 0} \frac{d(\gamma(t_0), \gamma(t_0+h))}{h} = F(\dot{\gamma}(t_0)).$$

$\square$

We can formulate now one of the main results of this section.

**Theorem 7.5** *For any Lipschitz curve  $\gamma : [0, 1] \rightarrow M$ ,  $l(\gamma)$  equals  $L(\gamma)$ .*

*Proof.* For any subdivision  $t_0 = 0 < t_1 < t_2 < \dots < t_n = 1$  of  $[0, 1]$ , from Theorem 7.1 and Lemma 7.2, it follows that

$$d(c(t_i), c(t_{i+1})) = \int_{t_i}^{t_{i+1}} \frac{d}{dt} d(c(t_i), c(t)) dt \leq \int_{t_i}^{t_{i+1}} F(\dot{\gamma}(t)) dt.$$

By summing, it follows that

$$\sum_{i=0}^{n-1} d(c(t_i), c(t_{i+1})) \leq \int_0^1 F(\dot{\gamma}(t)) dt = L(\gamma). \quad (7.1)$$

On the other hand, by Theorem 7.1 and Lemma 7.4,

$$l(\gamma) = \int_0^1 \frac{d}{dt} l(\gamma|_{[0,t]}) dt \geq \int_0^1 F(\dot{\gamma}(t)) dt = L(\gamma). \quad (7.2)$$

The conclusion follows from the relations (7.1) and (7.2).  $\square$

It can be seen that the function  $\delta$  is a quasi-distance on  $C_N$ . It is clear that Lemma 4.1 holds for the quasi-distance function  $\delta$ . Thus  $\lim_{n \rightarrow \infty} \delta(x, x_n) = 0$  is equivalent to  $\lim_{n \rightarrow \infty} \delta(x_n, x) = 0$  for any sequence  $\{x_n\}$ . In the case where  $F$  is absolute homogeneous,  $\delta$  is a genuine distance function on  $C_N$ .

Let  $c : [0, a] \rightarrow C_N$  be a Jordan arc parametrized by arclength, i.e.,  $l(c|_{[0,t]}) = t$  for all  $t \in [0, a]$ , where  $l$  is given in (4.1). By definition we have

$$d(c(t_1), c(t_2)) \leq \delta(c(t_1), c(t_2)) \leq l(c|_{[t_1, t_2]}) = |t_1 - t_2| \quad (7.3)$$

for any  $t_1, t_2 \in [0, a]$ . This implies that  $c : [0, a] \rightarrow M$  is a Lipschitz map (with respect to  $d$ ) and hence  $c$  is differentiable for almost all  $t$ . We will prove that  $c$  is a unit speed curve, i.e.,  $F(\dot{c}(t)) = 1$  for almost all  $t$ .

**Lemma 7.6** *For almost all  $t$ ,  $F(\dot{c}(t)) = 1$ . Conversely, if  $F(\dot{c}(t)) = 1$  for almost all  $t$ , then  $c$  is parametrized by arclength.*

*Proof.* Suppose that  $c$  is differentiable at  $t_0 \in (0, a)$ . Since

$$\lim_{t \searrow t_0} \frac{d(c(t_0), c(t))}{t - t_0} = \lim_{t \searrow t_0} F\left(\frac{\exp_{c(t_0)}^{-1} c(t)}{t - t_0}\right) = F((d \exp_{c(t_0)}^{-1})_{O_{c(t_0)}} \dot{c}(t_0)) = F(\dot{c}(t_0)),$$

we get, by (7.3),  $F(\dot{c}(t_0)) \leq 1$ . Hence,  $F(\dot{c}(t)) \leq 1$  for almost all  $t$ . Since  $a = l(c) = \int_0^a F(\dot{c}(t)) dt$ , by Theorem 7.5, it results  $\int_0^a (1 - F(\dot{c}(t))) dt = 0$ . Thus,  $F(\dot{c}(t)) = 1$  for almost all  $t$ , since  $1 - F(\dot{c}(t)) \geq 0$ .  $\square$

The following two lemmas follow immediately from Propositions 2.2, 2.3 and Lemma 7.6.

**Lemma 7.7** *For almost all  $t$ , we have*

$$\dot{c}(t) = v^f(t)^+ = v^b(t)^-.$$

**Lemma 7.8** *Suppose that  $c(t)$  is differentiable at  $t = t_0$ . If  $d_N \circ c(t)$  is differentiable at  $t = t_0$ , then  $(d_N \circ c)'(t_0) = g_X(X, \dot{c}(t_0))$ , where  $X$  denotes the velocity vector of an  $N$ -segment to  $c(t_0)$ .*

**Remark 7.9** It is clear that  $(d_N \circ c)(t)$  is differentiable at  $t = t_0$  if  $c(t_0)$  is not a branch cut point and if  $c(t)$  is differentiable at  $t = t_0$ . Here a cut point  $c(t)$  is called a *branch cut point* if  $c(t)$  admits more than two sectors. It will be proved in Lemma 8.1 that there exist at most countably many branch cut points.

**Lemma 7.10** *If  $\{c(t_n)\}$ , where  $t_n \in (0, a]$ , is a sequence of points on the curve  $c$  convergent to  $c(0)$  (with respect to  $d$ ), then  $\lim_{n \rightarrow \infty} \delta(c(0), c(t_n)) = 0$ .*

*Proof.* Let  $\{t_{n_i}\}$  be any convergent subsequence of  $\{t_n\}$ . Since  $\lim_{n \rightarrow \infty} d(c(0), c(t_n)) = 0$  and  $c$  is continuous, we get  $d(c(0), c(t_\infty)) = 0$ , where  $t_\infty$  denotes the limit of  $\{t_{n_i}\}$ . Thus,  $c(0) = c(t_\infty)$  and  $t_\infty = 0$ . This implies that  $\lim_{n \rightarrow \infty} t_n = 0$ . By definition,  $\delta(x, c(t_n)) \leq l(c|_{[0, t_n]}) = t_n$ . Therefore,  $\lim_{n \rightarrow \infty} \delta(c(0), c(t_n)) = 0$ .  $\square$

**Lemma 7.11** *Let  $\{x_n\}$  be a sequence of cut points of  $N$  convergent to a cut point  $x$ . If all  $x_n$  lie in a common sector  $\Sigma_x$  at  $x$ , then  $\lim_{n \rightarrow \infty} \delta(x, x_n) = 0$ .*

*Proof.* For each  $n$ , let  $e_n : [0, a_n] \rightarrow C_N$  denote a unit speed Jordan arc joining from  $x$  to  $x_n$ . It is clear that  $c := e_1$  and  $e_n$  ( $n > 1$ ) are Jordan arcs emanating from the common cut point  $x$ . Suppose that  $e_n(0, a_n]$  and  $c(0, a]$ , where  $a := a_1$ , have no common point for some  $n > 1$ . Let  $\{\epsilon_i\}$  be a decreasing sequence convergent to zero. Since  $e_n(0, a_n]$  and  $c(0, a]$  have no common point, we get the subarc  $c_i$  (lying in  $\Sigma_x$ ) of the circle centered at  $x$  with radius  $\epsilon_i$  cut off by  $e_n$  and  $c$  for each  $i$ . Let  $\gamma_i$  denote an  $N$ -segment to an interior point of  $c_i$  for each  $i$ . Then, any limit  $N$ -segment of the sequence  $\{\gamma_i\}$  as  $i \rightarrow \infty$ , is an  $N$ -segment to  $x$  lying in  $\Sigma_x$ . This contradicts the definition of a sector. Therefore, there exists  $t_n \in (0, a_n]$  satisfying  $e_n = c$  on  $[0, t_n]$  for each  $n$ . From Theorem 3.6 and Lemma 7.10,  $\lim_{n \rightarrow \infty} \delta(x, c(t_n)) = 0$ . Hence, by the triangle inequality, it is sufficient to prove  $\lim_{n \rightarrow \infty} \delta(c(t_n), x_n) = 0$ . It is obvious that each sector at  $e_n(t)$  ( $t_n < t < a_n$ ) containing  $e_n(t, t + \delta)$  for small  $\delta > 0$  shrinks to an  $N$ -segment to  $x$  as  $n \rightarrow \infty$ . Therefore, by Lemma 7.8, there exists  $\epsilon_0 \in (0, 1)$  such that

$$(d_N \circ e_n)'(t) < -\epsilon_0$$

for almost all  $t \in (t_n, a_n)$  and for all  $n$ . By integrating the equation above, we get  $\delta(c(t_n), x_n) < \frac{1}{\epsilon_0}(d(N, c(t_n)) - d(N, x_n))$  for all  $n$ . Since  $\lim_{n \rightarrow \infty} d(N, c(t_n)) = d(N, x) = \lim_{n \rightarrow \infty} d(N, x_n)$ , we obtain  $\lim_{n \rightarrow \infty} \delta(c(t_n), x_n) = 0$ .  $\square$

**Lemma 7.12** *Let  $\{x_n\}$  be a sequence of cut points of  $N$  convergent to a cut point  $x$ . If there are no sectors at  $x$  that contain an infinite subsequence of the sequence  $\{x_n\}$ , then  $\lim_{n \rightarrow \infty} \delta(x, x_n) = 0$ .*

*Proof.* For each  $n$ , let  $\Sigma_n$  denote the sector at  $x$  containing  $x_n$ . Then, from the hypothesis of our lemma, the sequence  $\{\Sigma_n\}$  shrinks to an  $N$ -segment to  $x$ . By applying the argument for the pair  $x_n$  and  $c(t_n)$  in the proof of Lemma 7.11 to the pair  $x$  and  $x_n$ , we get a number  $\epsilon_0 \in (0, 1)$  satisfying  $\delta(x, x_n) < \frac{1}{\epsilon_0}(d(N, x) - d(N, x_n))$  for each  $n$ . Hence we obtain  $\lim_{n \rightarrow \infty} \delta(x, x_n) = 0$ .  $\square$

**Theorem 7.13** *Let  $N$  be a closed subset of a forward complete 2-dimensional Finsler manifold  $(M, F)$  and  $C_N$  the cut locus of  $N$ . Then, the topology of  $C_N$  induced from the intrinsic metric  $\delta$  coincides with the induced topology of  $C_N$  from  $(M, F)$ .*

*Proof.* It is sufficient to prove that for any  $x \in C_N$  and any sequence  $\{x_n\}$  of cut points of  $N$ ,  $\lim_{n \rightarrow \infty} \delta(x, x_n) = 0$  if and only if  $\lim_{n \rightarrow \infty} d(x, x_n) = 0$ . Since  $d(x, y) \leq \delta(x, y)$  for any  $x, y \in C_N$ , it is trivial that  $\lim_{n \rightarrow \infty} \delta(x, x_n) = 0$  implies  $\lim_{n \rightarrow \infty} d(x, x_n) = 0$ . Suppose that  $\lim_{n \rightarrow \infty} d(x, x_n) = 0$ . By assuming that there exist an infinite subsequence  $\{x_{n_i}\}$  of  $\{x_n\}$  and a positive constant  $\eta$  satisfying  $\delta(x, x_{n_i}) > \eta$  for any  $n_i$ , we will get a contradiction. We may assume that all  $x_{n_i}$  lie in a common sector  $\Sigma_x$  at  $x$  or each  $x_{n_i}$  is contained in a mutually distinct sector at  $x$ , by choosing a subsequence of  $\{x_{n_i}\}$  if necessary. From Lemmas 7.11 and 7.12, we get  $0 < \eta \leq \lim_{n \rightarrow \infty} \delta(x, x_{n_i}) = 0$ . This is a contradiction.  $\square$

## 8 Proof of the completeness with respect to the intrinsic metric $\delta$

Let  $\{x_n\}$  denote a forward Cauchy sequence of points in  $C_N$  with respect to  $\delta$ . Here, without loss of generality, we may assume that  $\delta(x_n, x_m) < \infty$  for all  $n < m$ , i.e., all  $x_n$  lie in a common connected component of  $C_N$ . Since  $d \leq \delta$ , the sequence is a forward Cauchy sequence with respect to  $d$ . The metric space  $(M, d)$  is forward complete, therefore there exists a unique limit point  $\lim_{n \rightarrow \infty} x_n =: q$ . Since  $\lim_{n \rightarrow \infty} d(x_n, q) = \lim_{n \rightarrow \infty} d(q, x_n) = 0$ , we may choose a positive integer  $n_1$  and the positive number  $\delta_0$  chosen in Section 3 for the cut point  $x := x_{n_1}$  so as to satisfy  $q \in B_{\delta_0}(x)$ . We fix the point  $x = x_{n_1}$ . Choose any small positive number  $\epsilon$  so as to satisfy

$$d(q, x) > 2\epsilon \quad (8.1)$$

and

$$B_\epsilon(q) \subset B_{\delta_0}(x) \quad (8.2)$$

and fix it. Since the sequence  $\{x_n\}$  is a forward Cauchy sequence with respect to  $\delta$ , we may choose a positive integer  $n_0 := n_0(\epsilon)$  in such a way that

$$\delta(x_{n_0}, x_{n_0+k}) < \frac{\epsilon}{2} \quad (8.3)$$

for all  $k > 0$  and

$$d(q, x_{n_0}) < \frac{\epsilon}{2}. \quad (8.4)$$

For each integer  $k \geq 1$ , let  $c_k : [0, a_k] \rightarrow C_N$  denote a unit speed Jordan arc joining  $x_{n_0}$  to  $x_{n_0+k}$ . By (8.1), we may assume that

$$a_k < \frac{\epsilon}{2}. \quad (8.5)$$

**Lemma 8.1** *For each  $k \geq 1$ ,  $c_k[0, a_k]$  is a subset of  $B_\epsilon(q)$ .*

*Proof.* It follows from Theorem 7.1, Lemmas 7.2 and 7.6 that  $d(q, c_k(t)) \leq d(q, x_{n_0}) + t$  for any  $t \in [0, a_k]$ . Since  $a_k < \frac{\epsilon}{2}$  by (8.5) and  $d(q, x_{n_0}) < \frac{\epsilon}{2}$  by (8.4), we obtain  $d(q, c_k(t)) < \epsilon$  for any  $t \in [0, a_k]$ . □

**Lemma 8.2** *There exists a sector  $\Sigma_{q_\epsilon}$  at a cut point  $q_\epsilon$ , which is not an endpoint of  $C_N$ , such that  $q \in \Sigma_{q_\epsilon}$  and  $d(q, q_\epsilon) = 2\epsilon$ . Hence there exists a sector  $\Sigma_{q_{\epsilon_1}}$  at a cut point  $q_{\epsilon_1}$  of  $N$ , which is not an endpoint of  $C_N$ , such that  $q \in \Sigma_{q_{\epsilon_1}} \subset \Sigma_{q_\epsilon}$  and  $d(q, q_{\epsilon_1}) = 2\epsilon_1$  for some  $0 < \epsilon_1 < \epsilon$ .*

*Proof.* Let  $c : [0, b] \rightarrow C_N$  denote a unit speed Jordan arc joining  $x$  to  $x_{n_0}$ . Let  $c(t_0)$  denote a point on the arc  $c$  with  $d(q, c(t_0)) = \epsilon$ . The existence of  $c(t_0)$  is clear, since  $d(q, c(0)) = d(q, x) > 2\epsilon$  and  $d(q, c(b)) = d(q, x_{n_0}) < \frac{\epsilon}{2}$  by (8.1) and (8.4). Let  $\Sigma_{c(t_0)}^+$  denote the sector at  $c(t_0)$  containing  $x_{n_0}$ . By Lemma 8.1,  $x_{n_0+k} \in \Sigma_{c(t_0)}^+$  for all  $k \geq 1$ . Hence, the point  $q$  is an element of the closure of  $\Sigma_{c(t_0)}^+$ . Since the closure of  $\Sigma_{c(t_0)}^+$  is a subset of  $\Sigma_{c(t)}^+$  for any  $t < t_0$ ,  $q$  is an element of  $\Sigma_{q_\epsilon}^+$ , where  $\Sigma_{c(t)}^+$  denotes the sector at  $c(t)$  containing  $x_{n_0}$  and  $q_\epsilon$  denotes a point on the arc  $c$  with  $d(q, q_\epsilon) = 2\epsilon$ . Hence the sector  $\Sigma_{q_\epsilon}^+$  has the required property. □

**Lemma 8.3** *Let  $\{\Sigma_n\}$  be a decreasing sequence of sectors (i.e.,  $\Sigma_1 \supset \Sigma_2 \supset \Sigma_3 \supset \dots$ ) such that each  $\Sigma_n$  is a sector at a cut point  $q_n$  of  $N$ . Suppose that  $\lim_{n \rightarrow \infty} q_n := q$  exists and  $q \in \Sigma_n$  for all  $n$ . If  $q_n$  is not an endpoint of  $C_N$  for all  $n$ , then  $q$  is a cut point of  $N$ .*

*Proof.* For each  $n$ , let  $\alpha_n$  and  $\beta_n$  denote the  $N$ -segments that form part of the boundary of  $\Sigma_n$ . Since  $q_n$  is not an endpoint of  $C_N$  for each  $n$ ,  $\alpha_n \neq \beta_n$  for each  $n$ . Suppose first that there exists a subsequence of  $\{\Sigma_n\}$  which does not shrink to a single  $N$ -segment. Then, there exist at least two  $N$ -segments to  $q$ . This implies that  $q$  is a cut point of  $N$ . Suppose next that the sequence shrinks to a single  $N$ -segment. Then,  $\{\alpha_n\}$  and  $\{\beta_n\}$  shrink to a common  $N$ -segment  $\gamma : [0, l] \rightarrow M$  to  $q = \gamma(l)$ . Let  $\tilde{\gamma} : [0, \infty) \rightarrow M$  denote the geodesic extension of  $\gamma$ . For any sufficiently large  $n$ ,  $\tilde{\gamma}$  intersect the  $N$ -segment  $\alpha_n$  or  $\beta_n$  at a point  $\tilde{\gamma}(l_n)$ ,  $l_n > l$ . Since  $\lim_{n \rightarrow \infty} \tilde{\gamma}(l_n) = \gamma(l) = q$ , and  $\tilde{\gamma}|_{[0, l_n]}$  is not an  $N$ -segment,  $q$  is a cut point of  $N$ . □

**Theorem 8.4** *Let  $N$  be a closed subset of a forward complete 2-dimensional Finsler manifold  $(M, F)$ . Then the cut locus  $C_N$  of  $N$  with the intrinsic distance  $\delta$  is forward complete.*

*Proof.* Let  $\{x_n\}$  denote a forward Cauchy sequence with respect to  $\delta$ . Then, there exists a unique limit  $\lim_{n \rightarrow \infty} x_n =: q$  with respect to  $d$ , since  $d \leq \delta$  and  $(M, d)$  is forward complete. By Lemma 8.2, there exists a decreasing sequence of sectors  $\Sigma_n$  at cut points  $q_n$  such that  $\lim_{n \rightarrow \infty} q_n = q$  with respect to  $d$  and none of  $q_n$  is an endpoint of  $C_N$ . Hence, by Lemma 8.3,  $q$  is a cut point of  $N$ . From Theorem 7.13, we obtain  $\lim_{n \rightarrow \infty} \delta(x_n, q) = 0$ . □

## 9 The proof of Theorem C

For each sector  $\Sigma$  at a cut point  $x$  of  $N$ , we define a number  $\mu(\Sigma)$  by

$$\mu(\Sigma) := \max\{g_X(X, Y), g_Y(Y, X)\},$$

where  $X$  and  $Y$  denote the velocity vectors at  $x$  of the two unit speed  $N$ -segments that form part of the boundary of  $\Sigma$ . It follows from Lemma 1.2.3 in [S] that  $\mu(\Sigma) < 1$  if  $X \neq Y$ . Recall that if a cut point  $x$  of  $N$  admits more than two sectors, then  $x$  is called a branch cut point (see Remark 7.9).

For each  $n = 1, 2, 3, \dots$ , let  $A_n$  denote the subset of  $C_N$  which consists of all branch cut points that admit three sectors  $\Sigma^i, i = 1, 2, 3$ , satisfying  $\mu(\Sigma^i) \leq 1 - \frac{1}{n}$ . It is clear that  $\bigcup_{n=1}^{\infty} A_n$  is the set of all branch cut points of  $C_N$ .

**Lemma 9.1** *For each  $n$ , the set  $A_n$  is locally finite. Hence, the set of all branch cut points is at most countable.*

*Proof.* By assuming that there exists a ball  $B_r(x), (0 < r < \infty)$  containing infinitely many elements  $z_\alpha$  of  $A_n$  for some  $n$ , we will get a contradiction. The set  $\{z_\alpha\}$  has an accumulation point  $z$ . The point  $z$  is also an element of  $A_n$ , since any two  $N$ -segments forming the three sectors  $\Sigma$  with  $\mu(\Sigma) \leq 1 - \frac{1}{n} < 1$  cannot shrink to a single  $N$ -segment. Let  $\{z_j\}$  denote a sequence of points of  $\{z_\alpha\}$  convergent to  $z$ . Without loss of generality, we may assume that all  $z_j$  are in a common sector at  $z$  or each  $z_j$  lies in a mutually distinct sector at  $z$ , by taking a subsequence if necessary. Thus, there exist at most two limit  $N$ -segments to  $z$  of the sequence of  $N$ -segments to  $z_j$ . This contradicts the fact  $z_j \in A_n$ . □

By Lemma 9.1, there exist at most countably many branch cut points, but we do not know if the closure  $A_N$  of  $\bigcup_{n=1}^{\infty} A_n$  is countable or not. Here, we choose a tree  $T \subset C_N \cap B_{\delta_0}(x)$ , where  $x$  is a cut point of  $N$  and  $\delta_0$  is the positive number chosen in Section 3. We define a subset  $T^b$  of  $T$  by

$$T^b := \{y \in A_N \cap T \mid y \text{ admits a sector having no branch cut points in } T\}.$$

**Lemma 9.2** *The set  $T^b$  is countable.*

*Proof.* For each element  $y \in T^b$ , there exists the subarc  $c_y$  of  $S_{\delta_0}(x)$  cut off by the sector at  $y$  that has no branch cut points. It is clear that  $c_{y_1} \cap c_{y_2} = \emptyset$  if  $y_1 \neq y_2$ . Since there exist at most countably many non-overlapping subarcs of  $S_{\delta_0}(x)$ , it follows that  $T^b$  is countable. □

**Theorem 9.3** *The set  $C_N \setminus C_N^e$  is a union of countably many Jordan arcs, where  $C_N^e$  denotes the set of all endpoints of  $C_N$ .*



*Proof.* By Lemmas 9.1 and 9.2, there exist at most countably many elements  $\{x_i \mid i = 1, 2, 3, \dots\}$  in  $T^b \cup (T \cap \bigcup_{n=1}^{\infty} A_n)$ . Since it is trivial to see that  $T$  consists of a unique Jordan arc if  $T$  has no branch cut points, we may assume that  $x_1$  is a branch cut point. For each  $x_i (i > 1)$ , let  $m_i : [0, a_i] \rightarrow T$  be the unique Jordan arc joining from  $x_1$  to  $x_i$ . Choose any  $q \in T \setminus \bigcup_{i=2}^{\infty} |m_i|$ , where  $|m_i| := m_i[0, a_i]$ . Let  $c : [0, b] \rightarrow T$  be the Jordan arc joining from  $x_1$  to  $q$ . If  $q$  is not an endpoint of the cut locus  $C_N$ ,  $c$  has an extension  $\tilde{c} : [0, \tilde{b}] \rightarrow T$ . Then,  $q \notin \bigcup_{i=2}^{\infty} |m_i|$  implies that  $\tilde{c}|_{(b, \tilde{b})}$  does not intersect any  $x_i$  and hence has no branch cut points. Let  $b_1 (< b)$  be the maximum number  $b_1$  with  $c(b_1) = x_j$  for some  $j$ . Then,  $q$  lies in a Jordan arc (without any branch cut points except  $x_j$ ) emanating from some  $x_j$ . At each  $x_i$  there exist at most countably many such Jordan arcs in  $T$ . Therefore,  $T \setminus (C_N^e \cup \bigcup_{i=2}^{\infty} |m_i|)$  is a union of countably many Jordan arcs. This implies that  $C_N \setminus C_N^e$  is a union of countably many Jordan arcs.  $\square$

**Remark 9.4** Recall that even in the Riemannian case, there are compact convex surfaces of revolution such that the cut locus of a point on the surface admits a branch cut points with infinitely many ramifying branches ([GS]).

A critical point of the distance function on a Finsler manifold is defined analogously to the Riemannian distance function (see [C]), i.e., a point  $q \in M \setminus N$  is called a *critical point* of the distance function  $d_N$  from  $N$  if for any tangent vector  $v$  at  $q$ , there exists an  $N$ -segment  $\gamma : [0, l] \rightarrow M$  to  $q = \gamma(l)$  such that  $g_{\dot{\gamma}(l)}(\dot{\gamma}(l), v) \leq 0$ . It is trivial that any critical point of  $d_N$  admits at least two  $N$ -segments, and hence any critical point is a cut point of  $N$ . Notice that the Gromov isotopy lemma ([C]) holds for the distance function  $d_N$ . The proof of the isotopy lemma for the distance function  $d_N$  is the same as the Riemannian case.

**Lemma 9.5** *Let  $c : [a, b] \rightarrow C_N$  be a unit speed Jordan arc. Suppose that  $c(t)$  and  $(d_N \circ c)(t)$  are differentiable at  $t = t_0 \in (a, b)$ . If  $c(t_0)$  is a critical point of  $d_N$ , then  $(d_N \circ c)'(t_0) = 0$ .*

*Proof.* By supposing that  $(d_N \circ c)'(t_0) \neq 0$ , we will get a contradiction. We may assume that  $(d_N \circ c)'(t_0) > 0$ , by reversing the parameter of  $c$  if necessary. From Proposition 2.2, it follows that

$$0 < (d_N \circ c)'(t_0) = g_X(X, \dot{c}(t_0)) \leq g_Y(Y, \dot{c}(t_0))$$

for the velocity vector  $Y$  at  $c(t_0)$  of any  $N$ -segment to  $c(t_0)$ . Here  $X$  denotes the velocity vector at  $c(t_0)$  of an  $N$ -segment that form part of the boundary of the sector  $\Sigma_{c(t_0)}^+$  at  $c(t_0)$ . Hence,  $g_Y(Y, \dot{c}(t_0)) > 0$  for the velocity vector  $Y$  at  $c(t_0)$  of any  $N$ -segment to  $c(t_0)$ . This contradicts the fact that  $c(t_0)$  is a critical point of  $d_N$ .  $\square$

**Lemma 9.6** *For each unit speed Jordan arc  $c : [a, b] \rightarrow C_N$ , there exists a measure zero subset  $\mathcal{E}$  of  $d_N \circ c[a, b]$  such that if  $(d_N \circ c)(t) \notin \mathcal{E}$ , then  $(d_N \circ c)'(t) \neq 0$ .*

*Proof.* It was proved in Lemma 3.2 of [ShT] that the Sard theorem holds for a continuous function of one variable, i.e., the set

$$\mathcal{E}_1 := \{d_N \circ c(t) \mid (d_N \circ c) \text{ is differentiable at } t \in (a, b) \text{ and } (d_N \circ c)'(t) = 0\}$$

is of measure zero. On the other hand,  $d_N \circ c$  is differentiable almost everywhere, since  $d_N \circ c$  is a Lipschitz function. Hence, the image  $\mathcal{E}_2$  of non-differentiable points of  $d_N \circ c$  by this Lipschitz function is of measure zero. Therefore, the set  $\mathcal{E} := \mathcal{E}_1 \cup \mathcal{E}_2 \cup \{d_N(c(a)), d_N(c(b))\}$  is of measure zero and satisfies the required properties.  $\square$

**Theorem 9.7** *Let  $N$  be a closed subset of a 2-dimensional Finsler manifold  $(M, F)$ . Then, there exists a subset  $\mathcal{E} \subset [0, \sup d_N)$  of measure zero such that for any  $t \in (0, \sup d_N) \setminus \mathcal{E}$ , the set  $d_N^{-1}(t)$  is a union of disjoint continuous curves. Furthermore, any point  $q \in d_N^{-1}(t)$  admits at most two  $N$ -segments. In particular,  $d_N^{-1}(t)$  is a union of finitely many disjoint circles if  $N$  is compact.*

*Proof.* Let  $C_N^b$  denote the set consisting of all branch cut points of  $N$ . Since  $C_N^b$  is at most countable by Lemma 9.1, the set  $\mathcal{E}^b := d_N(C_N^b)$  is of measure zero. By Theorem 9.3,  $C_N \setminus C_N^e = \bigcup_{i=1}^{\infty} |c_i|$ , where  $c_i : [a_i, b_i] \rightarrow C_N$  denotes a unit speed Jordan arc. Hence, by applying Lemmas 9.5 and 9.6 for each  $c_i$ , there exists a measure zero set  $\mathcal{E}_i \subset (d_N \circ c_i)[a_i, b_i]$  such that if  $(d_N \circ c_i)(t) \notin \mathcal{E}_i$ , then  $(d_N \circ c_i)'(t) \neq 0$ . Let  $C_N^c$  denote the set consisting of all endpoints of  $C_N$  admitting more than one  $N$ -segment. Since  $C_N^c$  is a countable set, the set  $\mathcal{E}^c := d_N(C_N^c)$  is of measure zero. Thus, the set  $\mathcal{E} := \bigcup_{i=1}^{\infty} \mathcal{E}_i \cup \mathcal{E}^b \cup \mathcal{E}^c$  is of measure zero. Choose any  $s \in (0, \sup d_N) \setminus \mathcal{E}$ , and fix it. Suppose that  $d_N^{-1}(s)$  contains a critical point  $q$  of  $d_N$ . If the point  $q$  is an endpoint of  $C_N$ , then the point is an endpoint admitting more than one  $N$ -segment. Hence  $s = d_N(q) \in \mathcal{E}^c$ , which contradicts our assumption  $s \notin \mathcal{E}$ . This implies that  $q \in C_N \setminus C_N^e = \bigcup_{i=1}^{\infty} |c_i|$  and the point is a critical point of  $d_N$  on some  $c_i[a_i, b_i]$ . Suppose that  $q = c_i(t_i)$  for some  $t_i \in [a_i, b_i]$ . Since  $s \notin \mathcal{E}_i$ ,  $(d_N \circ c_i)(t_i) \neq 0$ . This is a contradiction by Lemma 9.5, since  $q = c_i(t_i)$  is a critical point of  $d_N$ . Therefore, any point  $q \in M \setminus N$  with  $d_N(q) \in (0, \sup d_N) \setminus \mathcal{E}$  is not a critical point and admits at most two  $N$ -segments.  $\square$

## References

- [BCS] D. Bao, S. S. Chern, Z. Shen, *An Introduction to Riemann Finsler Geometry*, Springer, GTM **200**, 2000.
- [B] M. Berger, *A panoramic view of Riemannian Geometry*, Springer, 2003.
- [Bh] Richard L. Bishop, *Decomposition of cut loci*, Proc. Amer. Math. Soc. **65** (1) (1977), 133–136.
- [BM] H. Busemann and W. Mayer, *On the foundations of the calculus of variations*, Trans. Amer. Math. Soc. **49** (1941), 173–198.

- [C] J. Cheeger, *Critical points of distance functions and applications to geometry* Geometric Topology: recent developments (Montecatini Terme, 1990), 1-38, Lecture Notes in Math., 1504, Berlin, 1991.
- [GS] H. Gluck, D. Singer, *Scattering of geodesic fields. II*, Ann. Math. **110**(2) (1979), 205–225.
- [H] J. Hebda, *Metric structure of cut loci in surfaces and Ambrose’s problem*, J. Diff. Geom., **40** (1994), 621–642.
- [KTI] K. Kondo, M. Tanaka, *Total curvatures of model surfaces control topology of complete open manifolds with radial curvature bounded below: I*, Math. Ann. **351**(2) (2011), 251–256.
- [ITd] J. Itoh, M. Tanaka, *The Hausdorff dimension of a cut locus on a smooth Riemannian manifold*, Tohoku Math. J. **50**(4) (1998), 571–575
- [IT] J. Itoh, M. Tanaka, *The Lipschitz continuity of the distance function to the cut locus*, Trans. of AMS, **353** (1) (2000), 21–40.
- [LN] Y. Y. Li, L. Nirenberg, *The distance function to the boundary, Finsler geometry, and the singular set of viscosity solutions of some Hamilton-Jacobi equations*, Comm. Pure Appl. Math. **58**(1) (2005), 85–146.
- [S] Z. Shen, *Lectures on Finsler Geometry*, World Scientific, 2001.
- [ShT] K. Shiohama, M. Tanaka, *Cut loci and distance spheres on Alexandrov surfaces*, Séminaires & Congrès, Collection SMF No.1, Actes de la table ronde de Géométrie différentielle en l’honneur Marcel Berger (1996), 531–560.
- [SST] K. Shiohama, T. Shioya, and M. Tanaka, *The Geometry of Total Curvature on Complete Open Surfaces*, Cambridge tracts in mathematics **159**, Cambridge University Press, Cambridge, 2003.
- [WZ] R.L. Wheeden, A. Zygmund, *Measure and Integral*, Marcel Dekker, New York, 1977.

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